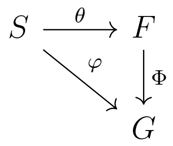
FREE GROUPS

SIDDHANT CHAUDHARY

Abstract. These notes contain a detailed discussion of free groups.

1. Free Groups

Definition 1.1. Let S be a set, and F be a group. Let $\theta : S \to F$ be a map. The pair (F, θ) is said to be a *free group* over S if any map $\varphi : S \to G$ can be extended to a *unique* homomorphism $\Phi : F \to G$ such that $\alpha = \Phi \circ \theta$, i.e the following diagram commutes:



Let us now prove some properties of free groups.

Theorem 1.2. Let (F, θ) be a free group on the set S.

(1) θ is one-one.
(2) $(F, \text{inclusion})$ is free on the set $\text{Im}(\theta)$.
(3) $F = \langle \text{Im}(\theta) \rangle$

Proof: Proving (1) is easy. Let G be an arbitrary group (say $\mathbb{Z}_2$), and let $\varphi : S \to G$ be any map. Suppose $\theta(s_1) = \theta(s_2)$ for some $s_1, s_2 \in S$. Since F is free on S, the map φ can be extended to a unique homomorphism, say Φ. Then we have

$$\Phi(\theta(s_1)) = \Phi(\theta(s_2))$$

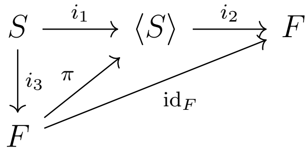
which implies that $\varphi(s_1) = \varphi(s_2)$. Since φ was an arbitrary map, it follows that $s_1 = s_2$, and hence θ is one-one.

To prove (2), consider the set $\text{Im}(\theta)$, and consider the inlusion map $\theta' : \text{Im}(\theta) \to F$, which is just the identity map. Let $\varphi : \text{Im}(\theta) \to G$ be any map, where G is any group. So, φ can be seen as a map from $S \to G$. Specifically, define $\varphi' : S \to G$ by $\varphi' = \varphi \circ \theta$. Since F is free on S, φ' can be extended to a unique homomorphism $\Phi : F \to G$ which satisfies $\varphi' = \Phi \circ \theta$. Now, it is easy to see that $\varphi = \Phi|_{\text{Im}(\theta)}$, so that $\varphi = \Phi \circ \text{inclusion}$. Uniqueness of the extension is easy to see. This proves the claim.

Now, we prove (3). By (2), we know that F is a free group over the set $\text{Im}(\theta)$, and hence we put $S = \text{Im}(\theta)$ for simplicity of notation.

Consider the following diagram, which we will show to be a commutative diagram:

Date: April 2020.

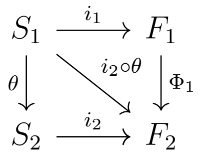


Here, $i_1 : S \to \langle S \rangle$, $i_2 : \langle S \rangle \to F$ and $i_3 : S \to F$ are inclusion maps. It is easy to see that $i_3 = i_2 \circ i_1$. Now, i_3 can be extended to a homomorphism from F to F, which by uniqueness is the id_F. Also, let π be the unique extension of i_1. Observe that $i_2 \circ \pi$ is another candidate homomorphism from $F \to F$, and by uniqueness we see that $i_2 \circ \pi = \mathrm{id}_F$, so that the diagram commutes. Finally, since id_F is surjective, it follows that i_2 is surjective, and hence proves that $F = \langle S \rangle$.

The next theorem shows that the free group over a set is unique. As a result, one can work with the group of *reduced words* over a set, which forms a free group (something that is not difficult to prove).

Theorem 1.3. Let F_1 be free on S_1 and let F_2 be free on S_2. Then, $F_1 \cong F_2$ if and only if $|S_1| = |S_2|$.

Proof: First, suppose S_1 and S_2 are in bijection (we don't assume them to be finite) and the proof will follow by the universal property. Consider the following commutative diagram (not hard to see why its commutative):



Here, i_1 and i_2 are inclusion maps, and θ is the bijection between S_1 and S_2. We have that

$$\Phi_1 \circ i_1 = i_2 \circ \theta$$

By a similar fashion, we can find a homomorphism Φ_2 from F_2 to F_1 such that

$$\Phi_2 \circ i_2 = i_1 \circ \theta^{-1}$$

Now, we show the following: we have that $\Phi_2\Phi_1$ is a homomorphism from F_1 to F_1. Also, we have

$$\Phi_2\Phi_1 \circ i_1 = \Phi_2 \circ (i_2 \circ \theta) = i_1 \circ \theta^{-1} \circ \theta = i_1$$

and hence $\Phi_1\Phi_2$ is an extension of the map $i_1 : S \to F_1$ to a homomorphism from F_1 to itself. By uniqueness of homomorphisms, it follows that $\Phi_2\Phi_1 = \mathrm{id}_{F_1}$. Similarly, we may prove that $\Phi_1\Phi_2 = \mathrm{id}_{F_2}$. So, this means that Φ_1 is an isomorphism.

For the converse, suppose $F_1 \cong F_2$. Abelianizing, we get

$$F_1/[F_1, F_1] \cong F_2/[F_2, F_2]$$

and this implies that

$$\mathbb{Z}(S_1) \cong \mathbb{Z}(S_2)$$

where $\mathbb{Z}(X)$ represents the free abelian group on X (see **Exercise** 2.4), or equivalently the free $\mathbb{Z}$-module over X. This means that $2\mathbb{Z}(S_1) \cong 2\mathbb{Z}(S_2)$, and hence taking quotients, we get

$$\mathbb{Z}(S_1)/2\mathbb{Z}(S_1) \cong \mathbb{Z}(S_2)/2\mathbb{Z}(S_2)$$

and using the fact that $\mathbb{Z}(X)/2\mathbb{Z}(X) \cong [\mathbb{Z}/2\mathbb{Z}](X)$ (need to verify this), we get

$$[\mathbb{Z}/2\mathbb{Z}](S_1) \cong [\mathbb{Z}/2\mathbb{Z}](S_2)$$

both of which are vector spaces. Comparing dimensions, we see that $|S_1| = |S_2|$, completing the proof.

(Need to verify that this proof is correct)

Definition 1.4. Suppose F is a free group over a set S. Then the cardinal number $|S|$ is said to be the *rank* of F.

The following is an abstract property of free groups:

Theorem 1.5. If G is any group, then G is isomorphic to the quotient group of some free group.

Proof: Let $S = G$, and consider the identity map from $S \to G$. The extension of this map to a map $\varphi : F(G) \to G$ is surjective, and by the first isomorphism theorem we have that $G \cong F(G)/\text{Ker}\varphi$. This proves the claim.

Finally, I will mention a result which is not trivial to prove (infact the proof is hard).

Theorem 1.6. Subgroups of free group are free.

2. Some Problems on Free Groups

Exercise 2.1. Show that free groups are torsion free (i.e there are no elements of finite order other than the identity). Infact, if $a^n = b^n$, show that $a = b$.

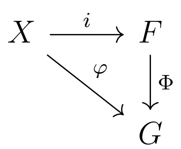
Solution:

Exercise 2.2. Show that in a free group, two commuting elements a, b must satisfy $a = c^u, b = c^v$ for some element c and some integers u, v. In particular, a free group has non-trivial center if and only if its rank is 1.

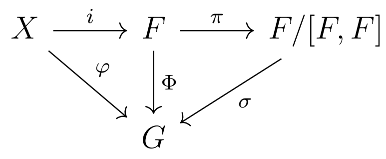
Exercise 2.3. If $w \neq 1$ in a free group, show that $C_F(w)$ (the centralizer) is an infinite cyclic group.

Exercise 2.4. If the free group F has rank n, then $F/[F,F] \cong \mathbb{Z}^n$, i.e abelianization of a free group leads to a free abelian group.

Solution: We will prove this not only for finite ranks, but arbitrary ranks. Let F be a free group over a set X. We know that $F/[F,F]$ is abelian. Let $\pi : F \to F/[F,F]$ be the natural projection map. Let G be any abelian group, and let $\varphi : X \to G$ be a set map. Then, there is a *unique* group homomorphism $\Phi : F \to G$ such that the diagram

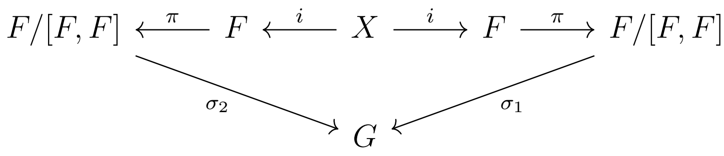


commutes. So, we see that $F/\text{Ker }\Phi \cong \Phi(F)$, and since $\Phi(F)$ is abelian, it follows that $[F,F] \subset \text{Ker }\Phi$ and hence Φ factors through $F/[F,F]$, so that the diagram



commutes, where σ is a homomorphism. Hence φ has been extended from X to $F/[F, F]$ (where the inclusion map is $\pi \circ i$).

We now prove uniqueness of the extension. Suppose σ_1, σ_2 are extensions of φ, so that the diagram



So we see that $\sigma_1 \circ \pi$ and $\sigma_2 \circ \pi$ are extensions of $\varphi : X \to G$ to F. By the universal property applied to F, we see that $\sigma_1 \circ \pi = \sigma_2 \circ \pi$, and since π is surjective, this implies $\sigma_1 = \sigma_2$. This completes the proof, showing that $F/[F, F]$ is the free abelian group over X.

3. Presentations

We can now formally define group presentations. First, some intuition. Suppose G is a group and $S \subset G$ such that $G = \langle S \rangle$. Let $F(S)$ be the free group over S. Then suppose G has presentation $G = \langle S|R \rangle$ where R represent the relations. Think of each relation as a word in $F(S)$, where each of these words has been collapsed to the identity. Let $\langle R^{F(S)} \rangle$ denote the normal closure of R in $F(S)$ (i.e, the smallest normal subgroup containing R, or equivalently the intersection of all normal subgroups containing R). It is then reasonable to define $G = F(S)/\langle R^{F(S)} \rangle$. And this is exactly how we do it.

So before formally defining presentations, let's do a quick lemma about normal closures:

Lemma: If $R \subseteq G$, then $\langle R^G \rangle = \langle grg^{-1} | g \in G, r \in R \rangle$

Proof: Let $H = \langle grg^{-1} | g \in G, r \in R \rangle$. It is clear that $H \leq \langle R^G \rangle$. We only need to show that H is a normal subgroup. To show this, let $g \in G$, and let

$$(g_1 r_1 g_1^{-1})...(g_n r_n g_n^{-1}) \in H$$

be an element of H. Then, we have that

$$g(g_1 r_1 g_1^{-1})...(g_n r_n g_n^{-1})g^{-1} = (gg_1 r_1 g_1^{-1} g^{-1})...(gg_n r_n g_n^{-1} g^{-1}) \in H$$

and this proves the lemma.

Definition 3.1. Let F be a free group over S. Let G be a group such that $G = \langle S \rangle$. A *presentation* for G is a pair (S, R) where $R \subset F$ such that $\langle R^G \rangle$ is the kernel of the homomorphism $\Phi : F(S) \to G$ that extends the identity map from $S \to G$, and in that case we have

$$G \cong F/\langle R^F \rangle$$

We now show that every group is presentable.

Theorem 3.2. All groups have presentations and finite groups have finite presentations.

Proof: Let $S = G$, and let $F(S)$ be the free group over S. Consider the identity mapping $\varphi : S \to G$. This extends to a homomorphism $\Phi : F(S) \to G$ such that $G \cong F(S)/\mathrm{Ker}(\Phi)$. Take any set R such that $\langle R \rangle = \mathrm{Ker}(\Phi)$. Then it follows that $G = \langle S | R \rangle$

For the second part, suppose G is finite.