ANA3, ASSIGNMENT-1

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1. Let S be a metric subspace of a metric space (X, d). We show that

$$A \textit{ is open in } (S,d) \iff A = S \cap U \textit{ where U is open in } X.$$

First, suppose A is open in (S, d). Then, for every $x \in A$, there is some $\delta_x > 0$ such that $B(x, \delta_x) \cap S \subset A$. Let

$$U = \bigcup_{x \in A} B(x, \delta_x)$$

so that U is open in (X, d), being a union of open sets. We show that $S \cap U = A$. First, suppose $x \in A$. Then, $x \in B(x, \delta_x) \cap S$, and hence $x \in S \cap U$, which shows that $A \subset S \cap U$. To show the reverse inclusion, suppose $x \in S \cap U$, which means that $x \in S$ and $x \in B(y, \delta_y)$ for some $y \in A$, meaning that $x \in B(y, \delta_y) \cap S$, and by definition of δ_y, we have that $B(y, \delta_y) \cap S \subset A$, implying $x \in A$, and hence $S \cap U \subset A$. So, $A = S \cap U$, proving one direction.

Conversely, let $A = S \cap U$ where U is open in (X, d). Let $x \in A$, so that $x \in S$ and $x \in U$. Since U is open, there is some $\delta > 0$ such that $B(x, \delta) \subset U$. This implies that $S \cap B(x, \delta) \subset S \cap U = A$, and hence this implies that A is open in (S, d), completing the proof.

Next, we show the following analogous statement:

$$A \textit{ is closed in } (S,d) \iff A = S \cap U \textit{ where U is closed in } X.$$

To prove this, suppose A is closed in S, so that $S \cap A^c$ is open in S. By what we have proved above, $S \cap A^c = S \cap U$ where U is some open subset of X. So,

$$A = (S \cap A^c)^c \cap S = (S \cap U)^c \cap S = (S^c \cup U^c) \cap S = (S^c \cap S) \cup (U^c \cap S) = S \cap U^c$$

and observe that U^c is open in X, because U is closed. Conversely, suppose $A = S \cap U$ for some closed subset U of X. We show that $S \cap A^c$ is open in S. Observe that

$$S \cap A^c = S \cap (S \cap U)^c = S \cap (S^c \cup U^c) = (S \cap S^c) \cup (S \cap U^c) = S \cap U^c$$

and by what we have shown above, $S \cap U^c$ is open in S, because U^c is open in X. This completes the proof.

2. Let (X, d) be a metric space, and we define

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

(i) We first show that d_1 is a metric on X. It is clear that $d_1(x, y)$ is non-negative, being a ratio of two non-negative reals. Observe that

$$d_1(x, x) = \frac{d(x, x)}{1 + d(x, x)} = 0$$

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Conversely, if $d_1(x,y) = 0$, then we have

$$\frac{d(x,y)}{1+d(x,y)} = 0$$

which implies that $d(x,y) = 0$, as the denominator is always positive, and hence $x = y$ since d is a metric. Next, we have

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d_1(y,x)$$

so that d_1 is symmetric in its arguments. Finally, we show the triangle inequality. Let $x, y, z \in X$. Consider the numbers $d(x,z)$, $d(y,z)$ and $d(x,y)$. First, suppose

$$\max\{d(y,z), d(x,y)\} \geq d(x,z)$$

and wlog suppose $d(y,z) \geq d(x,z)$. Then, we have

$$\begin{aligned}
& d(x,z) \leq d(y,z) \\
\implies & d(x,z) + d(x,z)d(y,z) \leq d(y,z) + d(x,z)d(y,z) \\
\implies & d(x,z)[1+d(y,z)] \leq d(y,z)[1+d(x,z)] \\
\implies & \frac{d(x,z)}{1+d(x,z)} \leq \frac{d(y,z)}{1+d(y,z)} \\
\implies & \frac{d(x,z)}{1+d(x,z)} \leq \frac{d(y,z)}{1+d(y,z)} + \frac{d(x,y)}{1+d(x,y)} \\
\implies & d_1(x,z) \leq d_1(x,y) + d_1(y,z)
\end{aligned}$$

In the second case, suppose

$$\max\{d(y,z), d(x,y)\} < d(x,z)$$

implying that $1 + d(y,z) < 1 + d(x,z)$ and $1 + d(x,y) < 1 + d(x,z)$. In that case, we have

$$\begin{aligned}
\frac{d(x,z)}{1+d(x,z)} &\leq \frac{d(x,y)+d(y,z)}{1+d(x,z)} \\
&= \frac{d(x,y)}{1+d(x,z)} + \frac{d(y,z)}{1+d(x,z)} \\
&< \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}
\end{aligned}$$

and in this case as well we have

$$d_1(x,z) \leq d_1(x,y) + d_1(y,z)$$

So in all cases, the triangle inequality holds and hence d_1 is a metric on X.

(ii) Here, we determine the class of all bounded sets in (X, d_1). We claim that all subsets B of X are bounded. To show this, let $B \subset X$, and fix $x_0 \in X$. Then, for all $x \in B$, we have

$$d(x_0,x) < 1 + d(x_0,x)$$

implying that

$$\frac{d(x_0,x)}{1+d(x_0,x)} < 1$$

for all $x \in B$, and hence

$$d_1(x_0,x) < 1$$

for all $x \in B$, showing that B is bounded. This completes the proof.

3. Let d_2, d_1 and d_{max} be the metrics in $\mathbb{R}^n$ associated to the $||\cdot||_2$, $||\cdot||_1$ and $||\cdot||_\infty$ norms respectively.

(i) We show that for any $x, y \in \mathbb{R}^n$

$$d_{\text{max}}(x,y) \le d_2(x,y) \le d_1(x,y) \le n d_{\text{max}}(x,y)$$

Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. So,

$$\begin{aligned} d_{\text{max}}(x,y)^2 &= \left(\max_{1\le i\le n} |x_i - y_i|\right)^2 \\ &= \max_{1\le i\le n} (x_i - y_i)^2 \\ &\le \sum_{i=1}^n (x_i - y_i)^2 \\ &= d_2(x,y)^2 \end{aligned}$$

and by taking square roots, it follows that

$$d_{\text{max}}(x,y) \le d_2(x,y)$$

Next, we have

$$\begin{aligned} d_2(x,y)^2 &= \sum_{i=1}^n |x_i - y_i|^2 \\ &\le \sum_{i=1}^n |x_i - y_i|^2 + 2 \sum_{1\le i<j\le n} |x_i - y_i||x_j - y_j| \\ &= \left(\sum_{i=1}^n |x_i - y_i|\right)^2 \\ &= d_1(x,y)^2 \end{aligned}$$

and by taking square roots, it follows that

$$d_2(x,y) \le d_1(x,y)$$

and hence we get

$$d_{\text{max}}(x,y) \le d_2(x,y) \le d_1(x,y)$$

Finally, we have

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^n |x_i - y_i| \\ &\le \sum_{i=1}^n \max_{1\le i\le n} |x_i - y_i| \\ &= \sum_{i=1}^n d_{\text{max}}(x,y) \\ &= n d_{\text{max}}(x,y) \end{aligned}$$

and this proves that

$$d_{\text{max}}(x,y) \le d_2(x,y) \le d_1(x,y) \le n d_{\text{max}}(x,y)$$

(ii) Let $x \in \mathbb{R}^n$ and let $r > 0$. We show that

$$B_1(x, r) \subseteq B_2(x, r) \subseteq B_{\max}(x, r) \subseteq B_1(x, rn)$$

Suppose $y \in B_1(x, r)$, so that $d_1(x, y) < r$, which implies that $d_2(x, y) < r$, and hence $y \in B_2(x, r)$. This shows the first inclusion. Next, if $y \in B_2(x, r)$, then $d_2(x, y) < r$, and hence $d_{\max}(x, y) < r$, implying that $y \in B_{\max}(x, r)$ and this shows the second inclusion. Finally, suppose $y \in B_{\max}(x, r)$, meaning that $d_{\max}(x, y) < r$. Since $d_1(x, y) \leq n d_{max}(x, y)$, this means that $\frac{d_1(x, y)}{n} < r$, and hence $d_1(x, y) < rn$, implying $y \in B_1(x, rn)$, showing the last inclusion. This completes the proof.

4. Let $X = (0, 1]$ and define

$$\gamma(x, y) = |x - y| + \left|\frac{1}{x} - \frac{1}{y}\right|$$

for $x, y \in X$.

(i) We show that γ is a metric on X. Clearly, $\gamma(x, y)$ is non-negative being a sum of two non-negative reals. Observe that

$$\gamma(x, x) = |x - x| + \left|\frac{1}{x} - \frac{1}{x}\right| = 0$$

and if $\gamma(x, y) = 0$, then

$$|x - y| + \left|\frac{1}{x} - \frac{1}{y}\right| = 0$$

which implies that $|x - y| = 0$, and hence $x = y$.

Next, we have

$$\gamma(x, y) = |x - y| + \left|\frac{1}{x} - \frac{1}{y}\right| = |y - x| + \left|\frac{1}{y} - \frac{1}{x}\right| = \gamma(y, x)$$

Finally, if $x, y, z \in X$, we have

$$\begin{aligned}\gamma(x, z) &= |x - z| + \left|\frac{1}{x} - \frac{1}{z}\right| \\ &\leq |x - y| + |y - z| + \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| \\ &= \gamma(x, y) + \gamma(y, z)\end{aligned}$$

where we just used the triangle inequality of the absolute value in $\mathbb{R}$. Hence, γ is a metric on X.

(ii) Next, we will show that $x_n \to x$ in the γ-metric if and only if $x_n \to x$ in the Euclidean metric.

First, suppose $x_n \to x$ in the γ metric, and observe that $x \neq 0$ (since $x \in X$). This means that

$$\gamma(x_n, x) \to 0$$

which means that

$$|x_n - x| + \left|\frac{1}{x_n} - \frac{1}{x}\right| \to 0$$

The above condition implies that

$$|x_n - x| \to 0$$

and hence $x_n \to x$ in the Euclidean metric.

Conversely, suppose $x_n \to x$ in the Euclidean metric, which means that

$$|x_n - x| \to 0$$

Now, the function

$$f(x) = \frac{1}{x}$$

is continuous in X, and hence this means that

$$|f(x_n) - f(x)| \to 0$$

which means

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| \to 0$$

So, we get

$$|x_n - x| + \left|\frac{1}{x_n} - \frac{1}{x}\right| \to 0$$

implying that $x_n \to x$ in the $\gamma-$metric. This completes the proof.

5. Let f be the function on $([0, \infty), d)$ defined by

$$f(x) = \begin{cases} 0 & , \text{ if } x \text{ is irrational} \\ \frac{1}{n} & , \text{ if } x = \frac{m}{n} \text{ with } \gcd(m, n) = 1 \end{cases}$$

(i) We show that f is a bounded function. Observe that for every $n \in \mathbb{N}$,

$$\frac{1}{n} \leq 1$$

and hence for every $x \in [0, \infty)$, we see that

$$|f(x)| \leq 1$$

implying that f is bounded.

(ii) We now show that f is continuous at each $x \notin \mathbb{Q}$ and discontinuous at each $x \in \mathbb{Q}$.

Suppose $x \in \mathbb{Q}$, so that $x = a/b$ with $\gcd(a, b) = 1$. Then, $f(x) = 1/b > 0$, since $b \in \mathbb{N}$. Every neighborhood of x contains an irrational number, and we know that f vanishes at every irrational number. This means that for every neighborhood N of x,

$$\sup_{y \in N \cap [0, \infty)} |f(y) - f(x)| \geq \frac{1}{b}$$

implying that f is not continuous at x.

Now, let $x \notin \mathbb{Q}$, so that $f(x) = 0$. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \epsilon$$

Take a small neighborhood $(x - \delta, x + \delta)$ of x such that $(x - \delta, x + \delta) \subset (0, \infty)$. Let S be the set of all rational numbers in $(x - \delta, x + \delta)$ in lowest terms such that the denominator of the rational is bounded above by N. Observe that S must be a finite set, because if not, the numerator will blow up, taking the number out of the interval. So, there is some $0 < \delta_1 < \delta$ such that all rational numbers in the interval $(x - \delta_1, x + \delta_1)$ in their lowest form have a denominator greater than N. This means that if $|y - x| < \delta_1$, then

$$|f(y) - f(x)| = |f(y)| < \epsilon$$

because $\frac{1}{N} < \epsilon$. Hence, f is continuous at x, i.e f is continuous at every irrational number.

6. Let $X = \mathbb{R}^2$, and let d denote the Euclidean metric on $\mathbb{R}^2$. Define

$$\rho(x,y) = \begin{cases} d(x,y) & , \quad \text{if } x,y \text{ are on the same ray from 0} \\ d(x,0)+d(0,y) & , \quad \text{otherwise} \end{cases}$$

(i) First, we show that ρ is a metric on X. It is clear that $\rho(x,y)$ is non-negative. Observe that

$$\rho(x,x) = d(x,x) = 0$$

and conversely, suppose $\rho(x,y) = 0$. If x,y lie on the same ray, then this means that $d(x,y) = 0$, and hence implies that $x = y$. If x and y don't lie on the same ray, then this means that

$$d(x,0) + d(0,y) = 0$$

implying that $d(x,0) = d(0,y) = 0$, and hence $x = y = 0$. Next, we have that

$$\rho(x,y) = \rho(y,x)$$

because the expressions defining ρ are symmetric in x and y. Finally, we show the triangle inequality. So, let $x,y,z \in X$. We will prove this using casework.

(1) In the first case, x and y lie on the same ray. Now, there are two possibilities for z. If z lies on the same ray as x and y, then we have

$$\begin{aligned}\rho(x,z) &= d(x,z) \\ &\le d(x,y) + d(y,z) \\ &= \rho(x,y) + \rho(y,z)\end{aligned}$$

If z lies on a different ray then x and y, then

$$\begin{aligned}\rho(x,z) &= d(x,0) + d(0,z) \\ &\le d(x,y) + d(y,0) + d(0,z) \\ &= \rho(x,y) + \rho(y,z)\end{aligned}$$

and hence in this case, the triangle inequality holds.

(2) In the second case, x and y lie on different rays. Again, there are two possibilities of z. First wlog suppose z lies on the same ray as x, then

$$\begin{aligned}\rho(x,z) &= d(x,z) \\ &\le d(x,0) + d(0,z) \\ &\le d(x,0) + d(0,y) + d(y,0) + d(0,z) \\ &= \rho(x,y) + \rho(y,z)\end{aligned}$$

Next, the other possibility is that all x,y,z lie on different rays. In that case,

$$\begin{aligned}\rho(x,z) &= d(x,0) + d(0,z) \\ &\le d(x,0) + d(0,y) + d(y,0) + d(0,z) \\ &= \rho(x,y) + \rho(y,z)\end{aligned}$$

and hence in this case as well, the triangle inequality holds.

So, ρ is a metric on X.

(ii) Next, we show that (X, ρ) is *not* separable. So, suppose D is a dense subset of this space. Consider the unit circle S^1, i.e

$$S^1 = \{(x, y) \in X | x^2 + y^2 = 1\}$$

It is clear that all points on S^1 lie on different rays, so if $p_1, p_2 \in S^1$, then

$$\rho(p_1, p_2) = d(p_1, 0) + d(0, p_2) = 2$$

so that the ρ-distance between any two points on S^1 is bounded below by $\dfrac{3}{2}$. Now, for every point in S^1, consider an open ball of radius $\dfrac{1}{2}$ (ball is taken with respect to the metric ρ). Since D is dense, each such ball contains a point of D, and no two such balls can contain the same point of D (because of the fact that ρ-distance between any two points on S^1 is bounded below by $\dfrac{3}{2}$). This proves that D has a subset in bijection with S^1, and we know that S_1 is uncountable. So, it follows that D is also uncountable, so that X is *not* separable.

7. Let (X, d) be a metric space, and let $A \subset X$.

(i) We show that

$$\overline{A} = \{x \in X | d(x, A) = 0\}$$

Suppose $x \in \overline{A}$, which means either $x \in A$ or x is a limit point of A. If $x \in A$, then $d(x, A) = \inf_{y \in A} d(x, y) = 0$. If x is a limit point of A, then there is a sequence $\{x_n\}$ of points of A such that $x_n \to x$. Again, this means that $d(x, A) = \inf_{y \in A} d(x, y) = 0$, and this shows that $\overline{A} \subset \{x \in X | d(x, A) = 0\}$.

To prove the reverse inclusion, suppose $x_0 \in \{x \in X | d(x, A) = 0\}$, meaning that $\inf_{x \in A} d(x_0, x) = 0$. If $x_0 \in A$, then $x_0 \in \overline{A}$. If $x_0 \notin A$, then the condition

$$\inf_{x \in A} d(x_0, x) = 0$$

implies the existence of a sequence $\{x_n\}$ of points of A such that $x_n \to x$, implying that x is a limit point of A, and hence $x \in \overline{A}$. This shows that $\{x \in X | d(x, A) = 0\} \subset \overline{A}$, hence showing the equality of the two sets.

(ii) Let $y \in X$ and $E \subset X$. We show that

$$y \text{ is not a limit point of } E \iff d(y, E \setminus \{y\}) > 0$$

First, suppose y is not a limit point of E, implying that there is some $\delta > 0$ such that

$$(B(y, \delta) \setminus \{y\}) \cap E = \phi$$

and hence $d(y, x) \geq \delta$ for all $x \in E \setminus \{y\}$, and taking the infimum over all $x \in E \setminus \{y\}$, we see that $d(y, E \setminus \{y\}) \geq \delta > 0$.

Conversely, suppose $d(y, E \setminus \{y\}) = \delta > 0$. This means that for any $x \in E \setminus \{y\}$, $d(y, x) \geq \delta$. Thus, it follows that

$$(B(y, \delta) \setminus \{y\}) \cap E = \phi$$

and hence y is not a limit point of E. This completes the proof.