ANA3 , ASSIGNMENT-2

SIDDHANT CHAUDHARY
BMC201953

Before solving the first problem, I will prove some lemmas that I am going to use.

Lemma 0.1. *Let $(X,d),(Y,\gamma)$ be two metric spaces, and define ρ on $X\times Y$ by*

$$\rho((x,y),(x',y')) = \max\{d(x,x'),\gamma(y,y')\}$$

Then ρ is a metric on $X\times Y$.

Proof. It is clear that ρ is a non-negative function on $X\times Y$. First, observe that $d(x,x)=0=\gamma(y,y)$, and hence this means that

$$\rho((x,y),(x,y)) = 0$$

for every $(x,y)\in X\times Y$. Conversely, suppose

$$\rho((x,y),(x',y')) = 0$$

for some $(x,y),(x',y')\in X\times Y$. This means that $d(x,x')=0=\gamma(y,y')$, and hence $(x,y)=(x',y')$. Next, we have

$$\rho((x,y),(x',y')) = \max\{d(x,x'),\gamma(y,y')\} = \max\{d(x',x),\gamma(y',y)\} = \rho((x',y'),(x,y))$$

for any $(x,y),(x',y')\in X\times Y$, showing that ρ is a symmetric function.

Finally, we show the triangle inequality. Let $(x,y),(x',y'),(x^*,y^*)$ be any three points in $X\times Y$. So, we have

$$\begin{aligned}\rho((x,y),(x^*,y^*)) &= \max\{d(x,x^*),\gamma(y,y^*)\}\\ &\le \max\{d(x,x')+d(x',x^*),\gamma(y,y')+\gamma(y',y^*)\}\\ &\le \max\{d(x,x'),\gamma(y,y')\}+\max\{d(x',x^*),\gamma(y',y^*)\}\\ &= \rho((x,y),(x',y'))+\rho((x',y'),(x^*,y^*))\end{aligned}$$

which shows that ρ is a metric on $X\times Y$. ■

Lemma 0.2. *Let X,Y be metric spaces as above, and give $X\times Y$ the metric ρ as above. Let $\pi : X\times Y\to X$ and $\pi' : X\times Y\to Y$ be usual projection maps. Then, π and π' are continuous.*

Proof. We will show that π is continuous, and the same proof will show that π' is also continuous. Let $U\subset X$ be an open set. Then, observe that $\pi^{-1}(U)=U\times Y$, and $U\times Y$ is an open subset of $X\times Y$, implying that π is continuous. ■

Lemma 0.3. *Let X,Y be metric spaces as above, and let (Z,d') be a metric space. Let $g : Z\to X\times Y$ be a mapping, and let $\pi : X\times Y\to X$, $\pi' : X\times Y\to Y$ be the projection maps (which are continuous by the previous lemma). Then, g is continuous if and only if $\pi\circ g : Z\to X$ and $\pi'\circ g : Z\to Y$ are continuous. In simpler words, g is continuous if and only if its components functions are continuous.*

Date: October 2020.

Proof. First, suppose g is continuous. By **Lemma** 0.2, we know that both projections π and π' are continuous. So, the compositions $\pi \circ g$ and $\pi' \circ g$ are both continuous. Conversely, suppose both compositions $\pi \circ g$ and $\pi' \circ g$ are continuous. Suppose Q is any open subset of $X \times Y$, and from basic topology we know that $Q = U \times V$, where $U \subset X$ and $V \subset Y$ are both open sets. So, we see that

$$g^{-1}(Q) = g^{-1}(U \times V) = (\pi \circ g)^{-1}(U) \cap (\pi' \circ g)^{-1}(V)$$

and since both $\pi \circ g$ and $\pi' \circ g$ are assumed to be continuous, we see that $(\pi \circ g)^{-1}(U) \cap (\pi' \circ g)^{-1}(V)$ is an open subset of Z (being an intersection of two open subsets), and hence this proves that g is continuous. ■

Lemma 0.4. *Let $(X, d), (Y, \gamma)$ be metric spaces, and suppose X is compact. Suppose $f : X \to Y$ is a continuous bijection. Then, the inverse map $f^{-1} : Y \to X$ is also continuous.*

Proof. Consider $f^{-1} : Y \to X$, and let U be a *closed* subset of X. We need to show that

$$(f^{-1})^{-1}(U) = f(U)$$

is closed in Y (the above equality is true because f is a bijection). Since X is compact and U is closed in X, U is also compact. Hence, $f(U)$ is compact, because f is continuous. Clearly, this shows that $f(U)$ is a closed subset of Y, and hence f^{-1} is continuous, completing the proof. ■

1. Let (X, d), (Y, γ) be metric spaces. On $X \times Y$ define the metric $\rho((x, y), (x', y')) = \max\{d(x, x'), \gamma(y, y')\}$ for $(x, y), (x', y') \in X \times Y$. Assume that X is compact. Let $f : X \to Y$ be a function. The *graph* of f is defined as the subset

$$G_f = \{(x, f(x)) : x \in X\}$$

of $X \times Y$. Show that f is continuous if and only if G_f is a compact subset of $X \times Y$.

Solution. By **Lemma** 0.1, we know that ρ is a metric on $X \times Y$.

First, suppose f is continuous. Consider the mapping $h : X \to X \times Y$ given by

$$h(x) = (x, f(x))$$

Since f is continuous, we see that the component functions of h are both continuous, and by **Lemma** 0.3, we see that h is a continuous function. Since X is compact, it follows that $h(X)$ is a compact subset of $X \times Y$. Clearly, $h(X) = G_f$, and hence this shows that G_f is a compact subset of $X \times Y$.

Conversely, suppose G_f is a compact subset of $X \times Y$, and consider the function h as above. h is clearly a bijection, and hence $h^{-1} : G_f \to X$ is a bijection as well. Moreover, it is easy to see that h^{-1} is just the projection $\pi|_{G_f}$ from G_f onto X, and by **Lemma** 0.2, we see that h^{-1} is a continuous map. Because G_f is compact, it follows that $(h^{-1})^{-1} = h$ is a continuous map by **Lemma** 0.4. Since f is a component function of h, and since h is continuous, we conclude that f is continuous by **Lemma** 0.3, and this completes the proof.

2. Let (X, d) be a connected metric space with atleast two points. Show that X is uncountable (Hint: consider sets of the form $\{y : d(x, y) < \delta\}$, $\{y : d(x, y) > \delta\}$ for appropriate values of δ.)

Solution. Fix a point $x_0 \in X$. Define the map $f : X \to \mathbb{R}$ given by $f(x) = d(x, x_0)$ for any $x \in X$. We know that the distance function is continuous, and hence $f : X \to \mathbb{R}$ is a continuous function. Moreover we know that X is connected, and since f is continuous, it follows that $f(X)$ is a connected subset of $\mathbb{R}$. Moreover, since X contains atleast two points, there is a point $x_1 \in X$ with $d(x_1, x_0) > 0$, i.e $f(X)$ contains the point 0 and a positive real number. So, $f(X)$ is a connected subset of $\mathbb{R}$ with atleast two points, and we know that the only such connected subsets of $\mathbb{R}$ are intervals (open, closed or half-open). Since any interval is uncountable, we conclude that $f(X)$ is uncountable. Finally, since $f : X \to f(X)$ is a surjective map, we conclude that X must be uncountable as well, completing the proof.

3. Let (X, d) be a compact metric space. For a set $A \subset X$, define the diameter of A by $d(A) = \sup\{d(a_1, a_2) : a_1, a_2 \in A\}$. Let $\{C_n\}$ be a sequence of non-empty closed sets such that $C_{n+1} \subset C_n$ for all $n \geq 1$, and that

$$\lim_{n\to\infty} d(C_n) = 0$$

What can you say about $C = \bigcap_{n=1}^{\infty} C_n$?

Solution. Since X is a compact space and each C_n is closed, it follows that each C_n is compact too. Moreover, the sequence $\{C_n\}$ forms a nested sequence of non-empty closed sets, it follows that this family of sets has the finite intersection property, i.e if $C_{i_1}, C_{i_2}, ..., C_{i_k}$ are any sets in this family with $i_1 < i_2 < ... < i_k$, then observe that

$$C_{i_1} \cap ... \cap C_{i_k} = C_{i_k} \neq \phi$$

by hypothesis. Since X is compact, this means that the intersection

$$\bigcap_{i=1}^{n} C_n \neq \phi$$

(this fact was proven in class). Next, we shall show that the intersections actually contains exactly one point. For the sake of contradiction, suppose the intersection contains two points, say x, x'. So, $x, x' \in C_n$ for every $n \in \mathbb{N}$. Also, let $\epsilon = d(x, x') > 0$, and by the definition of the diameter, we see that

$$d(C_n) \geq \epsilon$$

for all $n \in \mathbb{N}$, and hence

$$\lim_{n\to\infty} d(C_n) \geq \epsilon$$

contradicting the fact that the limit is zero. So, the intersection contains exactly one point, completing the proof.

4. Let (X, d) and (Y, d') be complete metric spaces. Define the metric ρ on $X \times Y$ by

$$\rho((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d'(y_1, y_2)\}$$

for $(x_1, y_1), (x_2, y_2) \in X \times Y$. Show that $(X \times Y, \rho)$ is a complete metric space.

Solution. By **Lemma** 0.1, we know that the metric ρ as defined makes $(X \times Y, \rho)$ a valid metric space. Now, suppose $\{p_n\}$ is a Cauchy sequence in $(X \times Y, \rho)$. Write

$$p_n = (x_n, y_n)$$

where $x_n \in X$ and $y_n \in Y$ for each $n \in \mathbb{N}$. Let $\epsilon > 0$ be given. Since $\{p_n\}$ is Cauchy, there is some $N \in \mathbb{N}$ such that

$$\rho((x_n, y_n), (x_m, y_m)) < \epsilon$$

for all $n, m \geq N$. By the way ρ is defined, this means that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$, and $d'(y_n, y_m) < \epsilon$ for all $n, m \geq N$. This means that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d) and (Y, d') respectively. Since they are complete by assumption, we conclude that $x_n \to x$ and $y_n \to y$ for some $x \in X$ and $y \in Y$. Again, let $\epsilon > 0$ be given. So, there are $N_1, N_2 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N_1$ and $d'(y_n, y) < \epsilon$ for all $n \geq N_2$. Put $N = \max\{N_1, N_2\}$, and we get that $d(x_n, x) < \epsilon$ and $d'(y_n, y) < \epsilon$ for all $n \geq N$. Putting this together, we see that $\rho((x_n, y_n), (x, y)) < \epsilon$ for all $n \geq N$, and hence $p_n \to p$, where $p = (x, y)$. So, this proves that $(X \times Y, \rho)$ is a complete metric space.

5. Show that the set of all irrational numbers in $\mathbb{R}$ is not the union of countably many closed nowhere dense sets.

Solution. For the sake of contradiction, suppose the set $\mathbb{Q}^c$ (the set of irrational numbers in $\mathbb{R}$) is the union of countably many closed nowhere dense sets, i.e we have

$$\mathbb{Q}^c = \bigcup_{i=1}^{\infty} C_n$$

where C_n is closed nowhere dense set for every $n \in \mathbb{N}$. Now, observe that

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$$

where the union on the RHS of the above equation is a countable union, since $\mathbb{Q}$ is a countable set. Moreover, for any $q \in \mathbb{Q}$, the set $\mathbb{R} - \{q\}$ is open and dense, implying that $\{q\}$ is a closed nowhere dense subset of $\mathbb{R}$. So, we see that

$$\mathbb{R} = \mathbb{Q}^c \cup \mathbb{Q} = \bigcup_{i=1}^{\infty} C_n \cup \bigcup_{q \in \mathbb{Q}} \{q\}$$

and observe that the RHS of the above equation is a *countable* union of closed nowhere dense sets. But, this clearly contradicts Baire's Theorem on $\mathbb{R}$ (and this was proven in class), since we know that $\mathbb{R}$ is complete. Hence, it follows that the set of irrational numbers is not the countable union of closed nowhere dense sets.