ANA3 , ASSIGNMENT-3

SIDDHANT CHAUDHARY
BMC201953

First I will prove some facts that I will use in some problems.

Lemma 0.1. *Let $\{f_n\}$, $\{g_n\}$ be two sequences in $(C[a,b],\rho)$ with the uniform metric ρ such that $f_n \to f$ uniformly and $g_n \to g$ uniformly, where $f,g \in C[a,b]$. Then*

(1) $f_n + g_n \to f + g$ *uniformly.*
(2) $cf_n \to cf$ *uniformly, for any* $c \in \mathbb{R}$.
(3) $f_n g_n \to fg$ *uniformly.*

Proof. Observe that saying $h_n \to h$ uniformly is the same as saying $\rho(h_n,h) \to 0$. So, our hypothesis is that $\rho(f_n,f) \to 0$ and $\rho(g_n,g) \to 0$ as $n \to \infty$. Also, since $\{f_n\}$ is a uniformly convergent sequence, it must be uniformly bounded, i.e $|f_n(x)| \leq M_1$ for all $n \geq 1$ and $x \in [a,b]$ for some $M_1 \geq 0$. Since g is continuous on $[a,b]$, $|g(x)| \leq M_2$ for some $M_2 \geq 0$.

(1) For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\rho(f_n+g_n, f+g) &= \sup_{x\in[a,b]} |f_n(x)+g_n(x)-f(x)-g(x)| \\
&\leq \sup_{x\in[a,b]} |f_n(x)-f(x)| + |g_n(x)-g(x)| \\
&\leq \sup_{x\in[a,b]} |f_n(x)-f(x)| + \sup_{x\in[a,b]} |g_n(x)-g(x)| \\
&= \rho(f_n,f) + \rho(g_n,g)
\end{aligned}$$

and the last term goes to 0 as $n \to \infty$. This means that $\rho(f_n+g_n, f+g) \to 0$ and hence $f_n + g_n \to f + g$ uniformly.

(2) This has a very similar argument as in (1).

(3) Observe that

$$\begin{aligned}
\rho(f_n g_n, fg) &= \sup_{x\in[a,b]} |f_n(x)g_n(x) - f(x)g(x)| \\
&= \sup_{x\in[a,b]} |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\
&\leq \sup_{x\in[a,b]} |f_n(x)|\,|g_n(x)-g(x)| + |g(x)|\,|f_n(x)-f(x)| \\
&\leq \sup_{x\in[a,b]} |f_n(x)|\,|g_n(x)-g(x)| + \sup_{x\in[a,b]} |g(x)|\,|f_n(x)-f(x)| \\
&\leq M_1\rho(g_n,g) + M_2\rho(f_n,f)
\end{aligned}$$

and the last term goes to 0 as $n \to \infty$. So, we have $\rho(f_n g_n, fg) \to 0$ and hence $f_n g_n \to fg$ uniformly.

■

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Lemma 0.2. *Let $\{f_n\}$ be a sequence in $(C[a,b]),\rho)$ such that $f_n \to f$ uniformly for some $f \in C[a,b]$. Then,*

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Proof. Since all functions in consideration are assumed to be continuous on $[a,b]$, they are all integrable on this interval, so the integrals exist. Now, let $\epsilon > 0$ be fixed, and let $N \in \mathbb{N}$ be such that

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$. So for any $n \geq N$, we have

$$\begin{aligned}\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| &= \left|\int_a^b f_n(x) - f(x)dx\right| \\ &\leq \int_a^b |f_n(x) - f(x)|dx \\ &\leq \int_a^b \epsilon dx \\ &= \epsilon(b-a)\end{aligned}$$

and since $\epsilon > 0$ is arbitrary, the claim holds. ■

1. Consider the space $C[0,1]$ with the uniform metric ρ. Let E, a subset of $C[0,1]$, and constants $M_1, M_2 > 0$ be such that $f \in E$ is differentiable on $(0,1)$, $|f(x)| \leq M_1$, $x \in [0,1]$, $f \in E$ and $|f'(x)| \leq M_2$, $x \in (0,1), f \in E$. Show that E has compact closure in $(C[0,1],\rho)$.

Solution. Put $S = [0,1]$, and we know that S is a compact metric space. By the **Arzela-Ascoli Theorem** (which has been proven in class), we know that a *closed* subset $E \subset C(S)$ is compact in $(C(S),\rho)$ if and only if E is uniformly bounded over S and E is uniformly equicontinuous over S. Also, we know that if a subset E of $C(S)$ is uniformly bounded and uniformly equicontinuous on S, then its *closure* in $(C(S),\rho)$ is also uniformly bounded and uniformly equicontinuous on S (and this has also been proven in class).

So, it is enough to show that the given set E is uniformly bounded and uniformly equicontinuous over S. By the given hypothesis, we know that $|f(x)| \leq M_1$ for each $f \in E$ and each $x \in S$, meaning that E is uniformly bounded over S. Let us now prove equicontinuity. Suppose $\epsilon > 0$ is fixed. Then, for any $x, y \in S$ and any $f \in E$ we see that

$$|x-y| < \epsilon/M_2 \implies |f(x) - f(y)| = |f'(c)||x-y| \leq M_2|x-y| \leq M_2\epsilon/M_2 = \epsilon$$

where the point c between x, y was furnished via the mean value theorem. In particular, this shows that E is uniformly equicontinuous over S, and this completes the solution.

2. Find a countable dense subset of $(C[0,1],\rho)$ where ρ denotes the uniform metric. Justify your answer.

Solution. By the **Weierstrass Approximation Theorem** (which has been proven in class), we know that the set of polynomials over $[0,1]$ is a dense subset of $(C[0,1],\rho)$. Now, we will show that *any* polynomial over $[0,1]$ can be *uniformly* *approximated* by a sequence of polynomials over $[0,1]$ with *rational coefficients*, and hence this will show that the set of polynomials over $[0,1]$ with *rational coefficients* (which clearly is a countable set) is dense in $(C[0,1],\rho)$, and that will show that $(C[0,1],\rho)$ is a separable space.

Now, we do another reduction. Let $\{f_n\}$, $\{g_n\}$ be sequences of real continuous functions over $[0,1]$ converging uniformly to f, g respectively. From **Lemma** 0.1 we know that

$$f_n + g_n \to f + g \quad \text{(uniformly)}$$
$$f_n g_n \to fg \quad \text{(uniformly)}$$

Clearly, these convergence properties can be extended to finite sums and products via induction. So, if

$$p(x) = a_k x^k + \ldots + a_0$$

is a polynomial over $[0,1]$, it is enough to show that for each $0 \leq i \leq k$, there is a sequence $\{q_{i,n}(x)\}$ of polynomials with *rational coefficients* such that

$$q_{i,n}(x) \to a_i x^i \quad \text{uniformly}$$

over $[0,1]$. Let $\{a_{i,n}\}$ be a sequence of rational numbers converging to a_i (possible because $\mathbb{Q}$ is dense in $\mathbb{R}$). Viewing this as a sequence of constant functions over $[0,1]$, we see immediately that

$$a_{i,n} \to a_i \quad \text{uniformly}$$

where we view a_i as a constant function on $[0,1]$. Now, put

$$r_{i,n}(x) = x^i$$

for each $n \in \mathbb{N}$. So, it is clear that

$$r_{i,n}(x) \to x^i \quad \text{uniformly}$$

So, define

$$q_{i,n}(x) = a_{i,n} r_{i,n}(x)$$

so that each $q_{i,n}(x)$ is a polynomial over $[0,1]$ with rational coefficients. Finally, by **Lemma** 0.1 (3), we see that

$$q_{i,n}(x) \to a_i x^i \quad \text{uniformly}$$

and hence this completes the proof. So, $C[0,1]$ is a separable space.

3. Let g be a continuous function on $[0,1]$ such that

$$\int_0^1 g(x) x^n dx = 0$$

for all $n \in \mathbb{N} \cup \{0\}$. Show that $g = 0$.

Solution. Let $p(x)$ be any polynomial over $[0,1]$. We first show that

$$\int_0^1 g(x) p(x) dx = 0$$

Suppose

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_0$$

where $a_0, a_1, ..., a_m \in \mathbb{R}$. Then, observe that

$$\begin{aligned}\int_0^1 g(x)p(x)dx &= \int_0^1 g(x)(a_0 + ... + a_m x^m)dx \\ &= \sum_{i=0}^{m} a_i \int_0^1 g(x)x^i dx \\ &= 0\end{aligned}$$

Now, we know that g is a continuous function on $[0,1]$. By the **Weierstrass Approximation Theorem**, we know that there is a sequence $\{P_n\}$ of polynomials over $[0,1]$ converging uniformly to g. Now if we consider the sequence

$$g_n(x) = g(x)$$

for each $n \in \mathbb{N}$, then clearly $g_n \to g$ uniformly. In that case, by **Lemma** 0.1 (3), we see that

$$P_n g_n \to g^2 \quad \text{(uniformly)}$$

and hence

$$P_n g \to g^2 \quad \text{(uniformly)}$$

and hence by **Lemma** 0.2 we get that

$$\lim_{n\to\infty} \int_0^1 P_n(x)g(x)dx = \int_0^1 g^2(x)dx$$

But as we showed above, the limit on the LHS of the above equation is zero, and hence

$$\int_0^1 g^2(x)dx = 0$$

implying that $g^2 = 0$ on $[0,1]$ (because g^2 is a positive continuous function), and hence we get that $g = 0$ on $[0,1]$.

4. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Show that there is a sequence of functions of the form

$$f_n(x,y) = \sum_{i=1}^{n} g_{i,n}(x)h_{i,n}(y)$$

for $x, y \in [0,1]$ where $g_{i,n}, h_{i,n}$ are continuous functions on $[0,1]$ such that $\{f_n\}$ converges uniformly to f.

Note: In the assignment, the functions f_n are of the form

$$f_n(x,y) = \sum_{i=1}^{n} g_i(x)h_i(y)$$

which make it look like the functions g_i, h_i for $i \geq 1$ are *independent* of n, which clearly changes the problem. However, I have confirmed with our professor that the g_i's and h_i's may depend on n, and that is why I have used the notation $g_{i,n}$ and $h_{i,n}$.

Solution. First, let A be the set of all functions of the form

$$\left\{\sum_{i=1}^{k} g_{i,k}(x)h_{i,k}(y) \mid k \in \mathbb{N}, g_{i,k}, h_{i,k} \in C[0,1]\right\}$$

Let us show that A is a subalgebra of $C([0,1]\times[0,1])$. By the *definition* of A, it is closed under addition. If $c \in \mathbb{R}$ is any constant, then

$$c\left(\sum_{i=1}^{k} g_{i,k}(x)h_{i,k}(y)\right) = \sum_{i=1}^{k}(cg_{i,k})(x)h_{i,k}(y)$$

and this shows that A is closed under scalar multiplication (because $C[0,1]$ is closed under scalar multiplication). Finally, to show that A is closed under multiplication, we have the following for any $k, l \in \mathbb{N}$.

$$\left(\sum_{i=1}^{k} g_{i,k}(x)h_{i,k}(y)\right)\left(\sum_{i=1}^{l} g_{i,l}(x)h_{i,l}(y)\right) = \sum_{i=1}^{k}\sum_{j=1}^{l} g_{i,k}(x)g_{i,l}(x)h_{i,k}(y)h_{i,l}(y)$$

and because $C[0,1]$ is closed under multiplication, it immediately follows that A is closed under multiplication. So, A is a subalgebra of $C([0,1]\times[0,1])$.

Next, we will show that A *separates* points on $[0,1]\times[0,1]$. So let $(x_1,y_1),(x_2,y_2) \in [0,1]\times[0,1]$ be two distinct points. Since these points are distinct, they differ in either their x-ordinate or their y-coordinate. Without loss of generality, suppose they differ in their x-coordinate (the proof when they differ in their y-coordinates is analogous), i.e

$$x_1 \neq x_2$$

Consider the function

$$\pi(x,y) = x$$

i.e the projection onto the first factor, which is evidently continuous. Moreover, it is easy to see that $\pi \in A$. So, we see that

$$\pi(x_1,y_1) \neq \pi(x_2,y_2)$$

and hence this shows that A separates points on $[0,1]\times[0,1]$, completing the proof.

Finally, it is easy to see that A contains all constant functions on $[0,1]\times[0,1]$. So, by the **Stone Weierstrass Theorem** applied to the compact set $[0,1]\times[0,1]$, it follows that A is dense in $C([0,1]\times[0,1])$.

Now, let $f : [0,1]\times[0,1] \to \mathbb{R}$ be a continuous function. By what we have proved above, there is a sequence $\{q_n\}$ of functions in A that converges uniformly to f over $[0,1]\times[0,1]$. Also, we know that

$$q_n(x,y) = \sum_{i=1}^{\phi(n)} g_{i,\phi(n)}(x)h_{i,\phi(n)}(y)$$

where $\phi : \mathbb{N} \to \mathbb{N}$ is some function, and each $g_{i,\phi(n)}, h_{i,\phi(n)} \in C[0,1]$. Now, suppose $\phi(1) = 1$, i.e

$$q_1(x,y) = g_{1,1}(x)h_{1,1}(y)$$

In this case, put $f_1 = q_1$. Next, suppose $\phi(1) = r > 1$, i.e

$$q_1(x,y) = g_{1,r}(x)h_{1,r}(y) + ... + g_{r,r}(x)h_{r,r}(y)$$

Here, put $f_i = 0+0+...+0$ (i times) for each $1 \leq i \leq r-1$, and put $f_r = q_1$ (by doing this, we are ensuring that the f_i contains exactly i terms for each $0 \leq i \leq r$ as required in the problem statement).

We repeat the above procedure inductively. Suppose we have found functions

$$f_1, ..., f_{n-1}$$

for some $n \in \mathbb{N}$ such that

$$f_j(x,y) = \sum_{i=1}^{j} g_{i,j}(x)h_{i,j}(y)$$

for each $1 \leq j \leq n-1$. Consider q_{n+1}. If $\phi(n+1) = n+1$, i.e if

$$q_{n+1}(x,y) = g_{1,n+1}(x,y)h_{1,n+1}(x,y) + ... + g_{n+1,n+1}(x,y)h_{n+1,n+1}(x,y)$$

then simply put

$$f_{n+1} = q_{n+1}$$

If $1 \leq \phi(n+1) < n+1$, then put

$$f_{n+1} = q_{n+1} + 0 + 0 + ... + 0 \quad (n+1-\phi(n+1)) \text{ zeroes}$$

Next, suppose $\phi(n+1) = n+1+r$ for some $r \in \mathbb{N}$. In that case, put

$$f_{n+i} = f_n + 0 + ... + 0 \quad i \text{ zeroes}$$

for each $1 \leq i \leq r$ and put

$$f_{n+1+r} = q_{n+1}$$

Then it is clear that the sequence $\{f_n\}$ converges to f uniformly (because $\{q_n\}$ does), and hence we have found the required sequence of functions.

5. Let $E = \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 = 1\}$, that is, E is the boundary of the open unit ball in $\mathbb{R}^2$. Note that E can be parametrized by the angle θ where $\tan(\theta) = y/x$ (as the tan function is periodic, $\theta = 0$ will be taken the same as $\theta = 2\pi$). A function $p : E \to \mathbb{R}$ is called a *trigonometric polynomial* if

$$p(\theta) = a_0 + \sum_{k=1}^{n}(a_k \cos(k\theta) + b_k \sin(k\theta))$$

where $a_j, b_j \in \mathbb{R}$. Show that any $\mathbb{R}$-valued continuous function on E can be uniformly approximated by trigonometric polynomials.

Solution. Observe that E is a compact subset of $\mathbb{R}^2$ being closed and bounded. To show that any $\mathbb{R}$-valued continuous function on E can be uniformly approximated by trigonometric polynomials, it is enough to show that the set of trigonometric polynomials is dense in $(C(E),\rho)$ (ρ is the uniform metric), and we will use the **Stone-Weierstrass Theorem** to do this (and this theorem was proved in class).

Let $T \subset (C(E),\rho)$ be the set of all trigonometric polynomials over E. Let us first show that T is a *subalgebra* of $C(E)$. So let p_1, p_2 be two trigonometric polynomials given by

$$p_1(\theta) = a_0 + \sum_{k=1}^{n}(a_k \cos(k\theta) + b_k \sin(k\theta))$$
$$p_2(\theta) = c_0 + \sum_{k=1}^{m}(c_k \cos(k\theta) + d_k \sin(k\theta))$$

and without loss of generality assume that $n \leq m$ (the case $n \geq m$ is analogous). So, we have the following.

$$(p_1+p_2)(\theta) = (a_0+c_0) + \sum_{k=1}^{n}((a_k+c_k)\cos(k\theta) + (b_k+d_k)\sin k\theta) + \sum_{k=n+1}^{m}(c_k\cos(k\theta) + d_k\sin(k\theta))$$

and this shows that T is closed under addition. Next, if $q_0 \in \mathbb{R}$, then

$$(q_0 p_1)(\theta) = q_0 a_0 + \sum_{k=1}^{n} (q_0 a_k \cos(k\theta) + q_0 b_k \sin(k\theta))$$

which shows that T is closed under scalar multiplication. Finally, we have

$$(p_1 p_2)(\theta) = \left[a_0 + \sum_{k=1}^{n} (a_k \cos(k\theta) + b_k \sin(k\theta)) \right] \left[c_0 + \sum_{k=1}^{m} (c_k \cos(k\theta) + d_k \sin(k\theta)) \right]$$

We can expand the above product, but observe that it is enough to show that each function of one of the forms $\cos(i\theta)\cos(j\theta)$, $\cos(i\theta)\sin(j\theta)$, $\sin(i\theta)\cos(j\theta)$ and $\sin(i\theta)\sin(j\theta)$ is a trigonometric polynomial for any positive integers $i, j \geq 1$. Then, by the fact that T is closed under addition and scalar multiplication, it will follow that T is closed under multiplication as well. Now, we have the usual trigonometric identities:

$$\begin{aligned}
\cos(i\theta)\cos(j\theta) &= \frac{1}{2}[\cos((i-j)\theta) + \cos((i+j)\theta)] \\
\cos(i\theta)\sin(j\theta) &= \frac{1}{2}[\sin((i+j)\theta) - \sin((i-j)\theta)] \\
\sin(i\theta)\cos(j\theta) &= \frac{1}{2}[\sin((i+j)\theta) + \sin((i-j)\theta)] \\
\sin(i\theta)\sin(j\theta) &= \frac{1}{2}[\cos((i-j)\theta) - \cos((i+j)\theta)]
\end{aligned}$$

which shows that each of these products is a trigonometric polynomial (which is easily seen by the definition of a trigonometric polynomial). Hence, it follows that T is closed under multiplication as well, and so T is a subalgebra of functions.

Next, it is clear that T contains all *constant functions* over E (because any constant function is a trigonometric polynomial as well). Finally, let us show that T *separates points* on E. Let $\theta_1, \theta_2 \in [0, 2\pi)$ be distinct numbers, and consider their corresponding points on the unit circle, i.e let

$$A = (\cos\theta_1, \sin\theta_1)$$
$$B = (\cos\theta_2, \sin\theta_2)$$

Since A and B are distinct points on the unit circle, they differ either in their x-coordinate or their y-coordinate. Without loss of generality, suppose that they differ in their x-coordinate (the proof being analogous if they differ in y-coordinate), i.e

$$\cos\theta_1 \neq \cos\theta_2$$

We know that

$$p(\theta) = \cos(\theta)$$

is a trigonometric polynomial, and we see that

$$p(\theta_1) \neq p(\theta_2)$$

and so we see that T separates points of E. So, by the **Stone Weierstrass Theorem**, it follows that T is a dense subset of $C(E)$, and this completes the proof.