ANA3 , ASSIGNMENT-4

SIDDHANT CHAUDHARY
BMC201953

Notation. As defined in class, the space H is

$$H := C([-\pi, \pi], \mathbb{C})$$

i.e the space of continuous complex valued functions over the interval $I := [-\pi, \pi]$. The inner product on this space is defined as

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$$

and it has been proven in class that this is a positive definite Hermitian product. Here are some results on Fourier Series which were proven in class and which I will be using in some problems.

Proposition 0.1. *Let $f \in H$ be a periodic function with period 2π, and let c_n be the Fourier coefficients for f (for $n \in \mathbb{Z}$). For any $N \geq 0$, define*

$$s_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

for any $x \in [-\pi, \pi]$. Then,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |s_N(x)|^2 dx = \sum_{n=-N}^{N} |c_n|^2$$

Proposition 0.2 (**Riemann-Lebesgue Lemma**). *Let $f \in H$, and let $\{c_n\}_{n\in\mathbb{Z}}$ be the Fourier coefficients of F. Then*

$$\lim_{|n|\to\infty} c_n = 0$$

and hence this means that

$$\lim_{|n|\to\infty} |c_n| = 0$$

Proposition 0.3. *Let $f \in H$, and let $\{c_n\}_{n\in\mathbb{Z}}$ be the Fourier coefficients of f. For any $N \geq 0$, let s_N be defined as in* **Proposition** 0.1*. Also, put*

$$c_0 = \frac{a_0}{2} \quad , \quad c_n = \frac{1}{2}(a_n - ib_n) \quad , \quad c_{-n} = \frac{1}{2}(a_n + ib_n)$$

for each $n \geq 1$, where a_0, a_n, b_n are complex numbers. Then for any $x \in [-\pi, \pi]$

$$s_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

Date: November 2020.

and so if $\{s_N\}_{N\in\mathbb{N}}$ converges uniformly to some function over $[-\pi,\pi]$, the series

$$\frac{a_0}{2}+\sum_{i=1}^{\infty}a_n\cos(nx)+b_n\sin(nx)$$

converges uniformly to the same function over $[-\pi,\pi]$.

Proof. This is an easy computation. For any $N\geq 0$ and $x\in[-\pi,\pi]$, we have the following chain of equations.

$$\begin{aligned}
s_N(x)&=c_0+\sum_{n=-N}^{-1}c_ne^{inx}+\sum_{n=1}^{N}c_ne^{inx}\\
&=c_0+\sum_{n=1}^{N}c_ne^{inx}+c_{-n}e^{-inx}\\
&=c_0+\sum_{n=1}^{N}c_n(\cos(nx)+i\sin(nx))+c_{-n}(\cos(nx)-i\sin(nx))\\
&=c_0+\sum_{n=1}^{N}(c_n+c_{-n})\cos(nx)+(ic_n-ic_{-n})\sin(nx)\\
&=\frac{a_0}{2}+\sum_{n=1}^{N}a_n\cos(nx)+b_n\sin(nx)
\end{aligned}$$

The rest of the statement follows easily from this. ■

1. Let $\{\varphi_n : n=0,1,...\}$ be an orthonormal system in H. Show that any finite subcollection of $\{\varphi_n\}$ is linearly independent.

Solution. Let $\{\varphi_{i_1},\varphi_{i_2},...,\varphi_{i_k}\}$ be any finite subcollection of the given system. Let us show that they are linearly independent. Suppose there are complex numbers $c_1,...,c_k$ such that

$$c_1\varphi_{i_1}+...+c_k\varphi_{i_k}=0$$

Fix $1\leq j\leq k$. Then we have

$$\langle c_1\varphi_{i_1}+...+c_k\varphi_{i_k},\varphi_{i_j}\rangle=\langle 0,\varphi_{i_j}\rangle=0$$

Also, by the bilinearity of this inner product, we see that

$$\begin{aligned}
\langle c_1\varphi_{i_1}+...+c_k\varphi_{i_k},\varphi_{i_j}\rangle&=\langle c_1\varphi_{i_1},\varphi_{i_j}\rangle+...+\langle c_k\varphi_{i_k},\varphi_{i_j}\rangle\\
&=c_1\langle\varphi_{i_1},\varphi_{i_j}\rangle+...+c_k\langle\varphi_{i_k},\varphi_{i_j}\rangle\\
&=c_j
\end{aligned}$$

where in the last step, we used the orthonormality of the given system. So, the above implies that $c_j=0$. Since $1\leq j\leq k$ was arbitrary, it follows that the subcollection $\{\varphi_{i_1},...,\varphi_{i_k}\}$ is linearly independent.

2. Let $f\in\mathscr{C}^1(\mathbb{R})$ be periodic with period 2π. Assume also that f is an odd function. Show that in the Fourier Series

$$\sum_{n=-\infty}^{\infty}c_ne^{inx}$$

of f, all the cosine terms will be absent.

Solution. Since $f \in \mathscr{C}^1(\mathbb{R})$ is periodic with period 2π, we have shown in class that the Fourier Series of f converges to f uniformly over $[-\pi, \pi]$. So, for any $x \in [-\pi, \pi]$, we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

As in **Proposition** 0.3 if we put

$$c_0 = \frac{a_0}{2} \quad , \quad c_n = \frac{1}{2}(a_n - ib_n) \quad , \quad c_{-n} = \frac{1}{2}(a_n + ib_n)$$

for every $n > 0$, then we see that for any $x \in [-\pi, \pi]$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Also, from the relations between c_n, a_n and b_n, we see that

$$a_n = c_{-n} + c_n$$

for every $n \geq 1$. Using the formula for the Fourier coefficients, we have

$$\begin{aligned} a_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)[e^{inx} + e^{-inx}] dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) 2\cos(nx) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \end{aligned}$$

Now, suppose f is an odd function, i.e $f(-x) = -f(x)$ for each $x \in [-\pi, \pi]$. Since cos is an even function, we see that for every $x \in [-\pi, \pi]$

$$f(-x)\cos(n(-x)) = -f(x)\cos(nx)$$

which means that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = 0$$

since the integrand above is an odd function. It follows that if f is an odd function, then the Fourier series of f does not have any cosine terms, which proves the claim.

3. Let $f \in \mathscr{C}^1(\mathbb{R})$ be periodic with period 2π. For $N = 0, 1, 2, .., $ let

$$s_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

Show that $f - s_N$ and s_N are orthogonal in H.

Solution. This is a straightforward computation. We have the following chain of equations.

$$\begin{aligned}
\langle f - s_N, s_N\rangle &= \int_{-\pi}^{\pi} [f(x) - s_N(x)]\overline{s_N(x)}dx \\
&= \int_{-\pi}^{\pi} f(x)\overline{s_N(x)}dx - \int_{-\pi}^{\pi} s_N(x)\overline{s_N(x)}dx \\
&= \int_{-\pi}^{\pi} f(x)\left(\sum_{n=-N}^{N} \overline{c_n}e^{-inx}\right) dx - \int_{-\pi}^{\pi} |s_N(x)|^2 dx \\
&= \int_{-\pi}^{\pi} \left(\sum_{n=-N}^{N} \overline{c_n}f(x)e^{-inx}\right) dx - \int_{-\pi}^{\pi} |s_N(x)|^2 dx \\
&= \sum_{n=-N}^{N} \int_{-\pi}^{\pi} \overline{c_n}f(x)e^{-inx}dx - \int_{-\pi}^{\pi} |s_N(x)|^2 dx \\
&= \sum_{n=-N}^{N} \overline{c_n} \int_{-\pi}^{\pi} f(x)e^{-inx}dx - \int_{-\pi}^{\pi} |s_N(x)|^2 dx \\
&= \sum_{n=-N}^{N} 2\pi c_n\overline{c_n} - \int_{-\pi}^{\pi} |s_N(x)|^2 dx \\
&= 2\pi \sum_{n=-N}^{N} |c_n|^2 - 2\pi \sum_{n=-N}^{N} |c_n|^2 \\
&= 0
\end{aligned}$$

where in the second last step, we used **Proposition** 0.1. This completes the proof.

4. Is

$$\sum_{n=1}^{\infty} \sin(nx) \quad x \in [-\pi, \pi]$$

the Fourier Series of some $f \in H$?

Solution. Suppose there was some $f \in H$ with the Fourier series as given above. As in **Proposition** 0.3, we have the formulae

$$c_0 = \frac{a_0}{2} \quad , \quad c_n = \frac{1}{2}(a_n - ib_n) \quad , \quad c_{-n} = \frac{1}{2}(a_n + ib_n)$$

where the Fourier series is

$$\frac{a_0}{2} + \sum_{i=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Now from the given Fourier series, we see that

$$a_n = 0$$

for all $n \in \mathbb{Z}$, and $b_n = 1$ for all $n \geq 1$. So, it follows that

$$c_0 = 0 \quad , c_n = \frac{-i}{2} \quad , c_{-n} = \frac{i}{2} \ \forall n \geq 1$$

So, it follows that

$$\lim_{|n|\to\infty} |c_n| = \frac{i}{2} \neq 0$$

which contradicts the **Riemann-Lebesgue Lemma** 0.2, since it was assumed that $f \in H$. So, there is no $f \in H$ having this Fourier series.

5. Find all the Fourier coefficients of the function

$$f(x) = \sin^2(3x) + \cos(7x) \quad , x \in [-\pi, \pi]$$

Solution. First, observe that $f \in \mathscr{C}^1(\mathbb{R})$ (and infact it is $\mathscr{C}^\infty(\mathbb{R})$, but that is not needed). It is also easily seen that f is periodic with period 2π. So, we have shown in class that the series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

converges uniformly to f over $[-\pi, \pi]$, where the Fourier Coefficients c_n are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

Now we can either directly compute c_n using the above formula, but there is a much easier way. Observe that by putting

$$c_0 = \frac{a_0}{2} \quad , \quad c_n = \frac{1}{2}(a_n - ib_n) \quad , \quad c_{-n} = \frac{1}{2}(a_n + ib_n)$$

we can write the series in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

and we can simply read off the coefficients a_n, b_n. Now, we know that for any $x \in \mathbb{R}$

$$\sin^2(3x) + \cos(7x) = \frac{1 - \cos(6x)}{2} + \cos(7x) = \frac{1}{2} - \frac{1}{2}\cos(6x) + \cos(7x)$$

So, it follows that

$$a_0 = 1 \quad , \quad a_6 = \frac{-1}{2} \quad , \quad a_7 = 1$$

and $b_n = 0$ for all ≥ 1, and all the rest of a_n's are 0 as well. Consequently, it follows that

$$c_0 = \frac{1}{2}$$
$$c_6 = \frac{-1}{4} = c_{-6}$$
$$c_7 = \frac{1}{2} = c_{-7}$$

and the rest of the c_n's are zero. So, all the Fourier coefficients have been found.