ANA3 , ASSIGNMENT-5

SIDDHANT CHAUDHARY
BMC201953

1. Let $X = [1, \infty)$, with the usual distance. The map T is given by

$$T(x) = \frac{1}{2}\left(x + \frac{2}{x}\right) \quad , \quad x \in X$$

Is T a contraction mapping on a complete metric space? Justify your answer.

Solution. Define the function

$$h(x, y) = \left|1 - \frac{2}{xy}\right|$$

for $x, y \geq 1$. It is easy to see that the maximum value of h over the set $[1, \infty) \times [1, \infty)$ is 1, because for any $x, y \geq 1$,

$$|xy - 2| \leq |xy|$$

Now, let $x, y \in \mathbb{R}$. So, we have

$$\begin{aligned} |T(x) - T(y)| &= \frac{1}{2}\left|(x - y) + 2\left(\frac{1}{x} - \frac{1}{y}\right)\right| \\ &= \frac{1}{2}\left|(x - y) - 2\left(\frac{x - y}{xy}\right)\right| \\ &= \frac{1}{2}|x - y|\left|1 - \frac{2}{xy}\right| \\ &\leq \frac{1}{2}|x - y| \end{aligned}$$

and this implies that T is indeed a contraction mapping. Now it is easy to see that $[1, \infty)$ is a complete metric space, because any Cauchy sequence will converge in this space, as any Cauchy sequence will be a subset of a compact subset of $[1, \infty)$, and hence it will be convergent. So, T is a contraction mapping on a complete metric space. ■

2. Show that the contraction mapping theorem need not hold in the following cases.

(1) (X, d) is not a complete metric space.
(2) (X, d) is a complete metric space, $f : X \to X$ satisfies

$$d(f(x), f(y)) \leq rd(x, y) \quad , \quad x, y \in X$$

where $r \geq 1$.

Date: November 2020.

Solution. For (1), let (X, d) be the space $\mathbb{R} - \{0\}$ with the usual distance. Clearly, X is not complete. Consider the map $T : X \to X$ defined by

$$T(x) = \frac{1}{2}x$$

Then, we have for any $x, y \in X$,

$$|T(x) - T(y)| = \frac{1}{2}|x - y| < \frac{3}{4}|x - y|$$

so that T is a contraction mapping. However, it is clear that T does not have any fixed point. This is the required counterexample.

For (2), let $X = \{0, 1\}$ be the discrete metric space where the metric d is defined by $d(0, 1) = 1$. It is clear that (X, d) is a complete metric space. Define the map $T : X \to X$ by $T(0) = 1$ and $T(1) = 0$. Then, for any $x, y \in X$ we have

$$d(T(x), T(y)) \leq d(x, y)$$

i.e $r = 1$. It is clear that T has no fixed points, and this is the required counterexample. ■

3. Let $A > 0, x_0 \in \mathbb{R}$. Let $g : [0, A] \to \mathbb{R}$ be a continuous function, and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipchitz continuous function. Show that the initial value problem for the ODE

$$x'(t) = g(t) + f(x(t)) \quad , \quad t \in (0, A)$$
$$x(0) = x_0$$

has a unique solution.

Solution. The proof is very similar to the proof of **Picard's Theorem** that we have done in class.
Because f is a Lipchitz continuous function, there is some $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in \mathbb{R}$. Now, let $\alpha > 0$ be a number such that $\alpha K < 1$. Also, there is some natural number $N \in \mathbb{N}$ such that $(N - 1)\alpha < A \leq N\alpha$. So, we will consider the N intervals $[0, \alpha], [\alpha, 2\alpha], ..., [(N - 1)\alpha, A]$.

First, consider the space $(C[0, \alpha], \rho^0)$ where ρ^0 is the uniform metric. We know that this is a complete metric space. Define the map $T^0 : C[0, \alpha] \to C[0, \alpha]$ by

$$T^0(y)(t) := x_0 + \int_0^t g(s)ds + \int_0^t f(y(s))ds \quad , \quad t \in [0, \alpha]$$

for any $y \in C[0, \alpha]$. The fact that T^0 is indeed a map from $C[0, \alpha]$ to itself is a consequence of the fundamental theorem of calculus, because we know that g, f and y are all continuous functions on $[0, \alpha]$. Now, we show that T^0 is a *contraction mapping* on this space. Observe that for any $y, z \in C[0, \alpha]$ and $t \in [0, \alpha]$

we have

$$\begin{aligned}|T^0(y)(t) - T^0(z)(t)| &= \left|\int_0^t f(y(s)) - f(z(s))ds\right| \\ &\leq \int_0^t |f(y(s)) - f(z(s))|ds \\ &\leq K\int_0^t |y(s) - z(s)|ds \\ &\leq K\rho^0(y,z)\int_0^t ds \\ &\leq K\alpha\rho^0(y,z)\end{aligned}$$

where in the last step, we used the fact that $t \leq \alpha$. Taking the supremum over $t \in [0,\alpha]$ in the RHS of the above equation, we get that

$$\rho^0(T^0(y), T^0(z)) \leq K\alpha\rho^0(y,z)$$

and because $K\alpha < 1$, this implies that T^0 is a *contraction mapping*. So, by the **Contraction Mapping Theorem**, there is some $y^0 \in C[0,\alpha]$ such that $T^0(y^0) = y^0$, i.e

$$y^0(t) = x_0 + \int_0^t g(s)ds + \int_0^t f(y^0(s))ds \quad , \quad t \in [0,\alpha]$$

Now we will repeat the same process as above with the interval $[\alpha, 2\alpha]$, but this time our constant will be $y^0(\alpha)$ (earlier the constant was x_0). So, consider the space $(C[\alpha,2\alpha], \rho^1)$ where ρ^1 is the uniform metric, and we know that this space is complete. As before, define a map $T^1 : C[\alpha,2\alpha] \to C[\alpha,2\alpha]$ by

$$T^1(y)(t) = y^0(\alpha) + \int_\alpha^t g(s)ds + \int_\alpha^t f(y(s))ds \quad , \quad t \in [\alpha,2\alpha]$$

for any $y \in C[\alpha,2\alpha]$. By the same proof as above, it follows that T^1 is a contraction mapping on this space, and hence by the **Contraction Mapping Theorem**, there is some $y^1 \in C[\alpha,2\alpha]$ such that $T^1(y^1) = y^1$, i.e

$$y^1(t) = y^0(\alpha) + \int_\alpha^t g(s)ds + \int_\alpha^t f(y^1(s))ds \quad , \quad t \in [\alpha,2\alpha]$$

Note that $y^0(\alpha) = y^1(\alpha)$, i.e these functions are equal on the common endpoint between the intervals $[0,\alpha]$ and $[\alpha,2\alpha]$. Continuing the same procedure as above for each of the N intervals, we see that there are functions $y^0 \in C[0,\alpha]$, $y^1 \in C[\alpha,2\alpha]$,...., $y^{N-1} \in C[(N-1)\alpha, A]$ such that

$$y^k((k+1)\alpha) = y^{k+1}((k+1)\alpha)$$

for each $0 \leq k \leq N-2$ and

$$y^0(t) = x_0 + \int_0^t g(s)ds + \int_0^t f(y^0(s))ds \quad , \quad t \in [0, \alpha]$$

$$y^1(t) = y^0(\alpha) + \int_\alpha^t g(s)ds + \int_\alpha^t f(y^1(s))ds \quad , \quad t \in [\alpha, 2\alpha]$$

$$\dots\dots$$

$$y^{N-2}(t) = y^{N-3}((N-2)\alpha) + \int_{(N-2)\alpha}^t g(s)ds + \int_{(N-2)\alpha}^t f(y^{N-2}(s))ds \quad , \quad t \in [(N-2)\alpha, (N-1)\alpha]$$

$$y^{N-1}(t) = y^{N-2}((N-1)\alpha) + \int_{(N-1)\alpha}^t g(s)ds + \int_{(N-1)\alpha}^t f(y^{N-1}(s))ds \quad , \quad t \in [(N-1)\alpha, A]$$

Note that the N functions $y^0, y^1, ..., y^{N-1}$ agree on the common endpoints of the intervals $[0, \alpha], ..., [(N-1)\alpha, A]$. So, we can define a continuous function $x : [0, A] \to \mathbb{R}$ by these N functions piecewise, i.e

$$x(t) = y^k(t) \quad , \quad t \in [k\alpha, \min\{A, (k+1)\alpha\}]$$

for $0 \leq k \leq N-1$. It is clear that x is a continuous function. Moreover, by the above integral equations that the y^ks satisfy, it follows that

$$x(t) = x_0 + \int_0^t g(s)ds + \int_0^t f(x(s))ds \quad , \quad t \in [0, A]$$

So, it follows that

$$x(0) = x_0$$

Moreover, because g, f and x are all continuous, we can apply the fundamental theorem of calculus in the open interval $(0, A)$ to get

$$x'(t) = g(t) + f(x(t))$$

for any $t \in (0, A)$. So, x is the required solution to the given initial value problem.

The uniqueness of x easily follows from the uniqueness of the functions $y^0, y^1, ..., y^{N-1}$ as guaranteed by the **Contraction Mapping Theorem**, because given the solution x, we can obtain functions $y^0, ..., y^{N-1}$ by restricting x to the suitable subintervals. This completes the proof. ■