ANALYSIS 3

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These are my course notes for **ANALYSIS-3**. Throughout the document, the symbol ■ will stand for QED. Any statement in red was left unfinished, and they make good exercises to work on.

Contents

1. Metric Spaces

In this section, we will revise some properties of metric spaces, and introduce some new ones. This is *not* an introduction to these.

Date: August 2020.

First, some notation. Consider $\mathbb{R}^n$. The standard Euclidean distance will be denoted by d_2, and the associated norm is $||\cdot||_2$. d_1 will denote the distance whose associated norm is $||\cdot||_1$, i.e

$$d_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

where $\boldsymbol{x} = (x_1, ..., x_n)$ and $\boldsymbol{y} = (y_1, ..., y_n)$. Finally, d_∞ will denote the distance whose associated norm is $||\cdot||_\infty$. Given a metric space (X, d), we will be considering the topology *induced* by the metric d, i.e the topology which has its basis all the open balls centered at points of X.

We now look at some examples of metric spaces.

Example 1.1. Let X be a non-empty set. The discrete metric is given by

$$d(x, y) = \begin{cases} 1 & \text{, if } x \neq y \\ 0 & \text{, otherwise} \end{cases}$$

Observe that in this case, every set is open: for instance, take the open ball around a point with radius δ, where $\delta < 1$. This leads to the *discrete topology* on X.

Example 1.2. Let $C[0, 1]$ denote the space of all real valued continuous functions on the interval $[0, 1]$. Define

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

It is straightforward to check that this is indeed a metric, and is called the *uniform metric*.

Example 1.3. Take $X = C[0, 1]$. Define

$$\rho(f, g) = \int_0^1 |f(x) - g(x)| dx$$

This is a *metric*, because of the continuity assumption, i.e $d(f, g) = 0$ implies that $f = g$. If continuity were removed, it wouldn't be a metric anymore. For instance, one could take two functions non-zero at only finitely many points.

Next, we introduce the concept of a *basis* (or *base*). This is just like the notion of a basis in point-set topology.

Definition 1.1. Let X be a metric space, and let $\{B_\alpha\}$ be a collection of open subsets of X. Suppose, for any open set G in X and every $x \in G$, there is an α for which $x \in B_\alpha \subset G$. Then, the collection $\{B_\alpha\}$ is said to be a *base* for X. Members of a base are called *basic open sets*.

The following proposition is simple and provides an equivalent definition of a base.

Proposition 1.1. *A collection $\{B_\alpha\}$ of open subsets of X is a base if and only if every open set in X can be written as a union of members of this collection.*

Definition 1.2. Let $D \subset X$. D is said to be *dense* in X if $\overline{D} = X$. X is said to be *separable* if it has a countable dense subset.

Theorem 1.2. *Let X be a metric space. Then the following are equivalent.*

(1) *X is separable.*
(2) *X has a countable base.*
(3) *Every open cover of X has a countable subcover.*

Proof. (1) $\implies$ (2). Suppose X is separable, and let D be a countable dense subset. Let G be the family of all open balls of the form

$$G := \{B(x,r)|x \in D, r \in \mathbb{Q}^+\}$$

and G is clearly countable. Let O be any open subset of X, and let $p \in O$. Let $B(p,\delta)$ be such that $B(p,\delta) \subset O$ (possible because O is open). Now, either $p \in D$, or there is some $x_p \in D$ and some $\delta' \in \mathbb{Q}^+$ such that $p \in B(x_p,\delta') \subset B(p,\delta) \subset O$. This shows that G is indeed a countable base for X.

(2) $\implies$ (3). (To be completed) ■

1.1. **Continuous Functions.** We already know the definition of continuous functions in metric spaces. In fact, continuous functions might as well be defined purely in terms of open sets. This we state as a theorem, which is straightforward to prove.

Theorem 1.3. *Let (X,d) and (Y,d') be metric spaces, and let $f : X \to Y$ be a map. Then the following are equivalent.*

(1) *f is continuous.*
(2) *$f^{-1}(V)$ is open in X for every open set V in Y.*
(3) *$f^{-1}(V)$ is closed in X for every closed set V in Y.*

And, ofcourse, continuity might as well be formulated using sequences.

Example 1.4. Let (X,d) be a metric space. Let $A \subset X$ fixed non-empty set. For any $x \in X$ define

$$d(x,A) = \inf\{d(x,y)|y \in A\}$$

$d(x,A)$ is called the *distance* from x to A. We show that distance is a continuous function.

For any $z \in A$ and $y \in X$, we have

$$d(x,z) \leq d(x,y) + d(y,z)$$

and taking the infimum of both sides over all $z \in A$, we see that

$$d(x,A) \leq d(x,y) + d(y,A)$$

implying that

$$d(x,A) - d(y,A) \leq d(x,y)$$

We can reverse the roles of x, y, and hence we get

$$|d(x,A) - d(y,A)| \leq d(x,y)$$

so that this function is continuous. Infact, we can see that this is *uniformly continuous*.

Theorem 1.4. *Let E, F be disjoint closed subsets of X. Then the following hold.*

(1) *There is a continuous function $f : X \to [0,1]$ such that $E = f^{-1}(\{0\})$ and $F = f^{-1}s(\{1\})$.*
(2) *There are disjoint open subsets U, V such that $E \subset U$ and $F \subset V$.*
(3) *If $x \notin F$, then there are disjoint open subsets U, V such that $x \in U$ and $F \subset V$.*

Proof. (1) Consider the function $f : X \to [0,1]$ defined by

$$f(x) = \frac{d(x,E)}{d(x,E) + d(x,F)}$$

The continuity of the function follows from the fact that the distance function is continuous. Moreover, since E and F are closed, we have that $d(x,Q) = 0 \iff x \in Q$ where $Q \in \{E,F\}$. This shows that f is our desired function, i.e $f^{-1}(\{0\}) = E$ and $f^{-1}(\{1\}) = F$.

(2) Take $U = f^{-1}\left(\left[1, \frac{1}{3}\right)\right)$ and take $V = f^{-1}\left(\left(\frac{2}{3}, 1\right]\right)$ noting that the range of f is the space $[0,1]$. (3) Take $E = \{x\}$ in this case, and apply (2). This shows that metric spaces are Hausdorff. ∎

1.2. **Compactness.** Here we prove some simple statements regarding compactness. Most of these will be familiar.

Proposition 1.5. *Let X be a metric space.*

(1) *Every compact subset of X is closed and bounded.*
(2) *Any closed subset of a compact set is compact.*

Proof. (1) Let K be a compact subset of X. Fix a point $x_0 \in X$. Consider the open cover of K given by the sets $B(x_0, n)$ for $n \in \mathbb{N}$. Since K is compact, this admits a finite subcover, and hence K is bounded.

(2) is very easy to prove. ∎

Definition 1.3. Let $\{A_\alpha\}$ be a collection of non-empty subsets of X. This collection is said to have the *finite intersection property* if any finite sub-collection of this collection has non-empty intersection.

Theorem 1.6. *A metric space X is compact if and only if every collection of closed subsets of X having the finite intersection property has non-empty intersection.*

Proof. To be completed ∎

Definition 1.4. Let X be a metric space. X is said to have the *Bolzano-Weierstrass property* if every sequence in X has a convergent subsequence. This property is also referred to as *sequential compactness*.

Proposition 1.7. *X has the Bolzano-Weierstrass property iff. every infinite set in X has a limit point.*

Proof. To be completed. ∎

Now, we see the equivalence between compactness and sequential compactness in metric spaces.

Theorem 1.8. *A metric space X is compact if and only if every infinite sequence in X has a convergent subsequence.*

Proof. To be completed ∎

Theorem 1.9. *Let X, Y be metric spaces, and let $f : X \to Y$ be a continuous map. If X is compact, then $f(X)$ is a compact subset of Y.*

Proof. To be completed ∎

Theorem 1.10. *Let X, Y be metric spaces, and let $f : X \to Y$ be a continuous map. If X is compact, then f is uniformly continuous on X.*

Proof. To be completed ∎

1.3. **Completeness.** Completeness in metric spaces is a familiar notion. For instance, $\mathbb{R}$ is complete.

Definition 1.5. A metric space X is said to be *complete* if every Cauchy sequence in X is convergent.

Lemma 1.11. *Let X be a complete metric space, and let $E \subset X$ be non-empty. Then, E is complete if and only if E is a closed subset of X.*

Proof. First, suppose E is complete, and let $x_0 \in X$ be a limit point of E. So, there is a sequence $\{x_n\}$ of points in E such that $x_n \to x_0$. Since E is complete, it follows that $x_0 \in E$, and hence E is closed. Conversely, suppose E is closed, and take any Cauchy sequence in E. Since X is complete, this sequence is convergent, and since E is closed, the limit must be in E. This completes the proof. ■

We will see some non-trivial examples of complete metric spaces in the next section.

1.4. **Uniform Convergence.** Here, we will consider only functions taking values in $\mathbb{R}$.

Definition 1.6. Let S be any set, and let $f_n : S \to \mathbb{R}$ be a sequence of functions. We see that this sequence *converges uniformly* to some $f : S \to \mathbb{R}$ if given any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$|f_n(s) - f(s)| < \epsilon$$

for all $n \geq N$ and for all $s \in S$, i.e the N does not depend on s.

Proposition 1.12. *Suppose S is a metric space, and suppose $f_n \to f$ uniformly, where $f_n, f : S \to \mathbb{R}$. Additionally, suppose each f_n is continuous at some $s_0 \in S$. Then, f is also continuous at s_0. This shows that uniform limits of continuous functions are continuous.*

Definition 1.7. Let S be any set. Define

$$B(S) := \{f : S \to \mathbb{R} | f \text{ is bounded}\}$$

and define

$$\rho(f, g) = \sup_{s \in S}\{|f(s) - g(s)|\}$$

for $f, g \in B(S)$. If in addition S is a metric space, we define

$$C(S) := \{f \in B(S) | f \text{ is continuous}\}$$

which is a subset of $B(S)$.

Proposition 1.13. *Let S be any set.*

(1) *ρ as defined above is a metric on $B(S)$.*
(2) *$f_n \to f$ uniformly over S if and only if $\rho(f_n, f) \to 0$ as $n \to \infty$. Hence, uniform convergence of functions is the same as convergence in this space.*
(3) *If in addition S is a metric space, $C(S)$ is a closed subset of $B(S)$.*

Proof. The first two assertions are immediate. (3) is true because uniform limits of continuous functiosn are continuous. ■

Theorem 1.14. *Let S be a non-empty set.*

(1) *The space* $(B(S), \rho)$ *is complete.*
(2) *If* S *is a metric space, then* $(C(S), \rho)$ *is also a complete metric space.*

Proof. (1) Suppose $\{f_n\}$ is a Cauchy sequence in $B(S)$. This means that given any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$\rho(f_n, f_m) < \epsilon$$

for all $m, n \geq N$, which means

$$|f_n(s) - f_m(s)| < \epsilon$$

for all $n, m \geq N$ and $s \in S$. For a fixed s, this defines a Cauchy sequence in $\mathbb{R}$, which is convergent. So define

$$f(s) = \lim_{n \to \infty} f_n(s)$$

Now, let $\epsilon > 0$ be given. So, there is some $N \in \mathbb{N}$ such that

$$|f_n(s) - f_m(s)| < \epsilon$$

for all $s \in S$ and $m, n \geq N$. Letting $m \to \infty$, we see that

$$|f_n(s) - f(s)| < \epsilon$$

for all $n \geq N$ and $s \in S$, implying that $f_n \to f$ uniformly over S, and hence $\{f_n\}$ has limit f in $B(S)$. The fact that f is bounded is immediate.

(2) This is true because uniform limits of continuous functions are continuous. This completes the proof. ■

Remark 1.14.1. The above two spaces give us some interesting examples of complete spaces. For instance, putting $S = [0, 1]$ shows that $C[0, 1]$ is a complete space.

1.5. **Total Boundedness.** We will provide a characterisation of compact spaces using this.

Definition 1.8. Let X be a metric space. X is said to be totally bounded if for every $\epsilon > 0$, there are *finitely* many points $x_1, ..., x_n \in X$ such that

$$X \subset B(x_1, \epsilon) \cup ... \cup B(x_n, \epsilon)$$

Example 1.5. It is easy to see that any totally bounded space is bounded. However, the converse is not true in general. For instance, let X be any infinite set equipped with the discrete metric. Then X is a bounded space but it is not totally bounded.

Theorem 1.15. *A metric space* X *is compact if and only if it is totally bounded and complete.*

Proof. We know that compact sets are complete. Any compact set is also totally bounded by considering the open cover

$$\bigcup_{x \in X} B(x, \epsilon)$$

So, only the converse needs to be shown. We also know that for metric spaces, compactness is equivalent to the Bolzano-Weierstrass property. So, we will show that every infinite sequence in X has a convergent subsequence.

The idea is as follows: given a sequence in X, we will try to extract a Cauchy-subsequence, and since the space X is complete, this Cauchy sequence will have a limit, and that will complete the proof.

So, let $\{x_n\}$ be a sequence in X. Now, X can be covered by finitely many open balls of radius 1. So, there is atleast one ball which contains a subsequence $\{x_{n_{1,k}}\}_{k\in\mathbb{N}}$ of $\{x_n\}$. Let this ball be $B(c_1, 1)$ where $c_1 \in X$. Next, finitely many open balls of radius $\frac{1}{2}$ cover X, and hence there is atleast one ball which contains a subsequence $\{x_{n_{2,k}}\}_{k\in\mathbb{N}}$ of $\{x_{n_{1,k}}\}_{k\in\mathbb{N}}$. Let this ball be $B(c_2, 1/2)$. Continuing this way, for every $i \in \mathbb{N}$, there is some open ball of radius $\frac{1}{i+1}$ containing a subsequence $\{x_{n_{i+1,k}}\}_{k\in\mathbb{N}}$ of $\{x_{n_{i,k}}\}_{k\in\mathbb{N}}$, and let this ball be $B(c_{i+1}, 1/(i+1))$. Now, we form a sequence $\{y_n\}$ as

$$y_i = x_{n_{i,i}}$$

and it is not hard to see that $\{y_n\}$ is a subsequence of $\{x_n\}$ (details can be easily filled in). Moreover, for any $m \in \mathbb{N}$, we see that

$$y_n \in B(c_m, 1/m)$$

for all $n \geq m$, i.e $\{y_n\}$ is a Cauchy sequence. So, $\{y_n\}$ is convergent, and hence $\{x_n\}$ has a convergent subsequence, completing the proof. ■

Remark 1.15.1. The proof technique we used above is a variant of the *Cantor's diagonal argument*, which is used in a lot of proofs in different fields of mathematics.

1.6. **Continuous Extensions.** In this section, we will see when continuous maps on subspaces can be extended to the whole space. A natural setting to investigate this question is on dense sets.

Lemma 1.16. *Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$ be uniformly continuous. If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y.*

Proof. Let $\epsilon > 0$ be given. So, there is some $\delta > 0$ such that

$$d(x, y) < \delta \implies \rho(f(x), f(y)) < \epsilon$$

for all $x, y \in X$. Also, there is some $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \delta$$

for all $n, m \geq N$. So, it follows that for $n, m \geq N$,

$$\rho(f(x_n), f(x_m)) < \epsilon$$

and hence $\{f(x_n)\}$ is a Cauchy sequence in Y. ■

Theorem 1.17. *Let (X, d) and (Y, ρ) be metric spaces such that Y is complete. Let D be a dense subset of X, and let $f : D \to Y$ be a uniformly continuous function. Then, there is a unique extension of f on X, i.e there is a unique continuous function F such that $F|_D = f$. Moreover, F is uniformly continuous.*

Before proving this theorem, by virtue of the following example, we see why uniform continuity *is* required.

Example 1.6. Let $X = [0,1]$, and let $D = (0,1)$. Let $Y = \mathbb{R}$, so that Y is complete. Consider the function

$$f(x) = \frac{1}{x}$$

on $(0,1)$. Clearly, this function cannot be extended to $[0,1]$. The reason is simple: f is not uniformly continuous.

Proof. (of **Theorem** 1.17). Let $x \in X \setminus D$, and let $\{x_n\}$ be a sequence of points in D converging to x (true because D is dense). So, $\{x_n\}$ is a Cauchy sequence in D, and by **Lemma** 1.16, we see that $\{f(x_n)\}$ is a Cauchy sequence in Y. Since Y is complete, this sequence is convergent, and define

$$f(x) = \lim_{n\to\infty} f(x_n)$$

First, we need to verify that this is well defined, i.e the limit is the same regardless of the sequence chosen. So, let $\{u_n\}$ and $\{w_n\}$ be two sequences in D converging to $x \in X \setminus D$. Let $\epsilon > 0$ be given. So, there is some $\delta > 0$ such that

$$d(x,y) < \delta \implies \rho(f(x), f(y)) < \epsilon$$

for all $x, y \in D$. Also, there is some $N \in \mathbb{N}$ such that

$$d(u_n, w_n) < \delta$$

for all $n \geq N$. This shows that

$$\rho(f(u_n), f(w_n)) < \epsilon$$

for all $n \geq N$, hence implying that f is well defined. So, from here on, we denote this extension by F.

We now show the uniform continuity of F. Let $\epsilon > 0$ be given, and let $\delta > 0$ be such that

$$\rho(F(x), F(y)) < \epsilon/3$$

for all $x, y \in D$ such that $d(x,y) < \delta$. Now, let x_0, y_0 be arbitrary points of X such that

$$d(x_0, y_0) < \delta$$

and let $\{x_n\}, \{y_n\}$ be sequences in D converging to x_0 and y_0 respectively. So, by our definition, we have

$$F(x_0) = \lim_{n\to\infty} F(x_n)$$
$$F(y_0) = \lim_{n\to\infty} F(y_n)$$

Choose $N \in \mathbb{N}$ such that

$$\rho(F(x_0), F(x_N)) < \epsilon/3$$
$$\rho(F(y_0), F(y_N)) < \epsilon/3$$

and that

$$d(x_N, y_N) < \delta$$

which is possible because $d(x_0, y_0) < \delta$. Finally, we have

$$\rho(F(x_0), F(y_0)) \leq \rho(F(x_0), F(x_N)) + \rho(F(x_N), F(y_N)) + \rho(F(y_N), F(y_0)) < \epsilon$$

and hence F is uniformly continuous over X. Uniqueness of F follows easily because D is dense in X. ■

1.7. **Connected Spaces.** In this subsection, we will be considering *connected* metric spaces. However, this notion also extends to general topological spaces.

Definition 1.9. A metric space X is said to be *connected* if it cannot be a written as a disjoint union of two non-empty open sets.

The following proposition is easy to verify.

Proposition 1.18. *Let X be a metric space. Then the following hold.*

(1) *X is connected if and only if it cannot be written as a disjoint union of two non-empty closed sets.*
(2) *X is connected if and only if the only subsets of X which are both open and closed are ϕ and X.*

The following proposition shows that the structure of connected sets in $\mathbb{R}$ is relatively simple.

Proposition 1.19. *The only connected subsets of $\mathbb{R}$ (under the usual metric) are the intervals (open, closed and half-open).*

Proof. To be completed. ■

Theorem 1.20. *Let X, Y be metric spaces, and let $f : X \to Y$ be continuous. If X is connected, then $f(X)$ is also connected.*

Proof. To be completed. ■

Definition 1.10. Let X be a metric space. A continuous map $f : [0,1] \to X$ is called a *path* in X. X is said to be *path-connected*, if any two points of X are connected by a path.

Proposition 1.21. *A path connected metric space is connected.*

Proof. To be completed. ■

Proposition 1.22. *Let X be a metric space, and let E be a connected subspace of X. Then, $\overline{E}$ is also connected.*

Proof. To be completed. ■

1.8. **A discussion on the Cantor Set.** In this section, we will study some properties of the *Cantor Set*.

Definition 1.11. A subset A of a metric space X is said to be *nowhere dense* if its closure has empty interior, i.e

$$\text{Int}(\overline{A}) = \phi$$

The terminology used can be explained via the following proposition.

Proposition 1.23. *A is a nowhere dense subset of X if and only if $(\overline{A})^c$ is an open dense subset of X. So in some sense, A is rare in X.*

Proof. First, suppose A is a nowhere dense subset of X. Hence, Int$(\overline{A}) = \phi$. Now, take any $x \in X$, and any $\delta > 0$. So, $B(x, \delta) \subset \overline{A}$ is not possible, and hence $B(x, \delta)$ contains a point of $(\overline{A})^c$, proving that $(\overline{A})^c$ is an open dense subset of X.

Conversely, suppose $(\overline{A})^c$ is an open dense subset of X. So, for every $x \in X$ and every $\delta > 0$, $B(x, \delta) \subset \overline{A}$ is not possible, and hence Int$(\overline{A}) = \phi$, proving that A is nowhere dense. ■

Definition 1.12. Let F_1 be obtained from $[0, 1]$ by removing the interval $(1/3, 2/3)$, i.e $F_1 = [0, 1/3] \cup [2/3, 1]$. Similarly, let F_2 be obtained by further deleting middle thirds. And inductively, we define F_n. The *Cantor Set* is the intersection

$$F = \bigcap_{i=1}^{\infty} F_i$$

Proposition 1.24. *The Cantor Set F is a non-empty compact subset of $[0, 1]$.*

Proof. Observe that the collection $\{F_n\}$ has the finite intersection property. Moreover, each F_n is closed. Since $[0, 1]$ is compact, it follows by **Theorem** 1.6 that F is non-empty, and being a closed subset of $[0, 1]$, it is compact. ■

Proposition 1.25. *The Cantor Set is an uncountable nowhere dense set.*

Proof. To be completed. ■

1.9. **Baire's Category Theorem.** In this section, we shall prove *Baire's Theorem*.

Theorem 1.26. *Let X be a complete metric space. Then the following hold.*

(1) *The intersection of countably many open dense sets is non-empty.*
(2) *X is not the union of countably many closed nowhere dense sets.*

Proof. By **Proposition** 1.40, it is easily seen that (1) and (2) are equivalent by taking complements. So, we will just prove (1).

To prove (1), let $\{U_n\}_{n\in\mathbb{N}}$ be a collection of open dense subsets of X. Let $B(x_1, r_1)$ be an open ball in U_1 (possible as $U_1 \neq \phi$), where $r_1 < 1$. Since U_2 is dense, $U_2 \cap B(x_1, r_1) \neq \phi$, and moreover, $U_2 \cap B(x_1, r_1)$ is open. So, take an open ball $B(x_2, r_2) \subset U_2 \cap B(x_1, r_1)$ such that $r_2 < \frac{1}{2}$ and that

$$\overline{B(x_2, r_2)} \subset U_2 \cap B(x_1, r_1)$$

which is clearly possible. Observe that

$$\overline{B(x_2, r_2)} \subset B(x_1, r_1)$$

and that

$$B(x_2, r_2) \subset U_1 \cap U_2$$

Suppose we have chosen open balls $B(x_1, r_1), ..., B(x_n, r_n)$ such that

$$\overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1})$$
$$\overline{B(x_{n-1}, r_{n-1})} \subset B(x_{n-2}, r_{n-2})$$
$$\dots$$
$$\overline{B(x_2, r_2)} \subset B(x_1, r_1)$$

with $B(x_n, r_n) \subset U_1 \cap U_2 \cap ... \cap U_n$ and $r_n < \dfrac{1}{n}$. Consider U_{n+1}, which is open and dense, and hence $U_{n+1} \cap B(x_n, r_n)$ is a non-empty open set, and hence we can choose a ball $B(x_{n+1}, r_{n+1})$ such that $\overline{B(x_{n+1}, r_{n+1})} \subset U_{n+1} \cap B(x_n, r_n)$ with $r_{n+1} < \dfrac{1}{n+1}$. Clearly, it follows that

$$\overline{B(x_{n+1}, r_{n+1})} \subset B(x_n, r_n)$$

and

$$B(x_{n+1}, r_{n+1}) \subset U_1 \cap ... \cap U_{n+1}$$

So inductively, such a family $\{B(x_n, r_n)\}_{n\in\mathbb{N}}$ exists.

Observe that for a fixed $N \in \mathbb{N}$, $x_n, x_m \in B(x_N, r_N)$ for all $n, m \geq N$, and hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, this sequence converges to some point, say $x \in X$. Let $n > 1$ be fixed. So, we have that

$$x_m \in B(x_n, r_n)$$

for all $m \geq n$, and this means that

$$d(x_n, x_m) < \frac{1}{n}$$

for all $m \geq n$. Letting $m \to \infty$, we see that

$$d(x_n, x) \leq \frac{1}{n}$$

because the distance map is continuous, and this means that

$$x \in \overline{B(x_n, r_n)}$$

for all $n \geq 1$. But this means that $x \in B(x_n, r_n)$ for all $n \geq 1$, and this shows that

$$x \in \bigcap_{n=1}^{\infty} U_n$$

completing the proof. ∎

Along the lines of the above proof, we can prove a stronger result.

Theorem 1.27 (**Baire's Category Theorem**). *Let X be a complete metric space. Then the following hold and are equivalent.*

(1) *The intersection of countably many open dense sets in X is dense.*
(2) *Any non-empty open subset of X is never contained in the countable union of closed nowhere dense sets.*

Proof. Again, by **Proposition** 1.40, it is clear that (1) and (2) are equivalent. So, we will only prove (1). Let $z \in X$ be any point, and let $B(z, \delta)$ be any ball, where $\delta > 0$. Since U_1 is dense and open, $B(z, \delta) \cap U_1$ is a non-empty open set. So, there is some ball $B(x_1, r_1)$ such that

$$\overline{B(x_1, r_1)} \subset B(z, \delta) \cap U_1$$

Now proceed as in **Theorem** 1.26. So, we get a point $x \in B(x_1, r_1)$ such that

$$x \in \bigcap_{n=1}^{\infty} U_n$$

and clearly $x \in B(z, \delta)$, proving that the infinite intersection is dense in X. ∎

Corollary 1.27.1. *$\mathbb{Q}$ is not the intersection of countably many open sets in $\mathbb{R}$.*

Proof. If $q \in \mathbb{Q}$, then note that $\mathbb{R} \setminus \{q\}$ is an open dense subset of $\mathbb{R}$. Now, for the sake of contradiction, suppose

$$Q = \bigcap_{i=1}^{\infty} U_n$$

for open sets in $\mathbb{R}$. Observe that each U_n is dense in $\mathbb{R}$ (as $\mathbb{Q}$ is). This implies that

$$\bigcap_{i=1}^{\infty} U_n \cap \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\} = \phi$$

which contradicts **Baire's Theorem** 1.45. ∎

Theorem 1.28. *$\mathbb{R}$ cannot be written as a disjoint union of atleast two countably many closed non-empty sets.*

Proof. To be completed. ∎

Definition 1.13. A subset E of a space X is said to be of the *first category* if it is contained in some countable union of closed nowhere dense sets.

The following proposition is clear by **Baire's Theorem** 1.45.

Proposition 1.29. *Let X be a complete metric space. Then, no open subset of X is of the first category.*

1.10. **Applications of BCT.** First, we define notion of the *oscillation* of a function at a point, which intuitively measures to what extent a function is continuous.

Definition 1.14. Let U be a metric space, and let $g : U \to \mathbb{R}$ be a function. For any $x \in U$, define

$$\text{osc}_g(x) = o(g, x) = \lim_{\delta \to 0} \sup\{|g(y) - g(x)| : y \in B(x, \delta)\}$$

Remark 1.29.1. This is not the usual definition of oscillation (see the first example given below). Usually, it is defined as the quantity

$$\lim_{\delta \to 0} \sup\{|g(t) - g(s)| : t, s \in B(x, \delta)\}$$

Proposition 1.30. *Let $g : U \to \mathbb{R}$ be a function as above, and let $x \in U$. Then, g is continuous at x if and only if $o(g, x) = 0$.*

Proof. This follows by the definition of oscillation at a point. ∎

Example 1.7. In this example, we will show that the set

$$\{x : o(g, x) \geq r\}$$

is *not necessarily closed* (and this is not the case with the usual definition of oscillation). Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \begin{cases} 0 & , x \notin \mathbb{Q} \\ 1 & , x \in \mathbb{Q}, x \neq 0 \\ 1/2 & , x = 0 \end{cases}$$

Then, it is easily seen that $o(g, x) = 1$ for any $x \neq 0$, and that $o(g, 0) = 1/2$, which proves the claim by taking $r = 1$.

Lemma 1.31. *Let U be a metric space, and let $g : U \to \mathbb{R}$ be a function. Then for $\epsilon > 0$,*

$$\overline{A} = \overline{\{x \in U : o(g,x) \geq 2\epsilon\}} \subset \{x \in U : o(g,x) \geq \epsilon/2\} = B$$

Proof. If $x_0 \in A$, then it is clear that $x_0 \in B$. Next, suppose x_0 is a limit point of A. For the sake of contradiction, suppose

$$o(g, x_0) < \epsilon/2$$

So, there is some $\delta > 0$ such that for all $y \in B(x_0, \delta)$, we have

$$|f(y) - f(x_0)| < \epsilon/2$$

Now, let $x \in A$ such that $x \in B(x_0, \delta)$. Now, choose some $\delta' > 0$ such that

$$B(x, \delta') \subset B(x_0, \delta)$$

Next, observe that for any $y \in B(x, \delta')$, we have

$$|f(y) - f(x)| \leq |f(y) - f(x_0)| + |f(x) - f(x_0)| < \epsilon$$

which contradicts the fact that $x \in A$. So, it must be that $o(g, x_0) \geq \epsilon/2$, and this shows that $\overline{A} \subset B$, completing the proof. ∎

Theorem 1.32. *Let X be a complete space, and let $U \subset X$ be an open subset of X. Suppose $f_n : U \to \mathbb{R}$ is a sequence of real valued functions on U, and suppose $f_n \to f$ pointwise. Then, the set of discontinuities of f is of the first category.*

Proof. To be completed. ∎

1.11. **Towards Arzela-Ascoli.** Let S be a compact metric space. By **Theorem 1.14**, we know that the space $(C(S), \rho)$ is complete, where ρ is the uniform metric. For this section, our goal will be to characterize compact subspaces of $(C(S), \rho)$.

Definition 1.15. Let E be a subset of $C(S)$.

(1) Let $x \in S$ be fixed. We say that E is *equicontinuous* at $x \in S$ if for each $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

for all $y \in B(x, \delta)$ and for all $f \in E$. So, intuitively, the family of functions in E are continuous at x where the number δ works for all members of the family.

(2) The set E is said to be *uniformly equicontinuous* if for each $\epsilon > 0$, there is some $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

for all $x, y \in S$ such that $d(x, y) < \delta$ and for all $f \in E$.

Remark 1.32.1. The best way to remember this notion is as follows. When we say a function is *uniformly continuous*, we can choose a δ that works for all *points in the domain*. When we say a family of functions is *equicontinuous*, we can find a δ that works for all the *functions in the family*. Finally, *uniform equicontinuity* is the combination of these two notions, i.e we can find a δ that works for all points in the domain *and* for all functions in the family.

Definition 1.16. As above, let E be a subset of $C(S)$.

(1) The set E is said to be *pointwise-bounded* if for each $x \in S$, there is some M_x such that
$$|f(x)| \leq M_x$$
for all $f \in E$.

(2) The set E is said to be *uniformly bounded* if there is some M such that
$$|f(x)| \leq M$$
for all $x \in S$ and $f \in E$, i.e the number M_x is independent of x.

The idea of uniform boundedness translates to boundedness in the space $C(S)$ as we see below.

Proposition 1.33. *Let $E \subseteq C(S)$. Then, E is uniformly bounded if and only if E is a bounded subset of $(C(S), \rho)$.*

Proof. First, suppose E is uniformly bounded. Then, there is some constant $M \geq 0$ such that
$$|f(x)| \leq M$$
for all $f \in E$ and $x \in S$. Let $0 : S \to \mathbb{R}$ be the zero function, which is in $C(S)$. Moreover, we have
$$\rho(0, f) = \sup_{x \in S} |f(x)| \leq M$$
for each $f \in E$, and hence $E \subset B(0, M+1)$, implying that E is a bounded subset of $C(S)$. Conversely, suppose E is a bounded subset of $C(S)$. So, there is some element $f' \in C(S)$ and some $M > 0$ such that
$$\rho(f', f) = \sup_{x \in S} |f'(x) - f(x)| < M$$
for all $f \in E$. Since f' is continuous and S is compact, let $N > 0$ be such that $|f'(x)| \leq N$ for all $x \in S$. So for all $f \in E$ and $x \in S$, we have
$$|f'(x) - f(x)| < M$$
and hence
$$|f(x)| < |f'(x)| + M \leq N + M$$
implying that E is uniformly bounded. ■

1.12. **Arzela-Ascoli Theorem.** First, let me describe the idea behind this theorem. As before, assume S is a compact metric space, and consider the space $(C(S), \rho)$. As in basic compactness arguments, we know that any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. We try to ask a similar question in $C(S)$; given any bounded sequence (which in this case, we use the notion of pointwise/uniform boundedness), is there a convergent subsequence? Recall that convergence in $(C(S), \rho)$ is equivalent to uniform convergence of a sequence of functions. The Arzela-Ascoli Theorem will answer this question precisely.

First, we mention a fairly simple lemma, which has nothing to do with the above discussion.

Lemma 1.34. *Let S be a compact space, and let $\{f_n\}$ be a pointwise bounded sequence in $(C(S), \rho)$. Let E be a countable subset of S. Then, there is a subsequence of $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}(x)\}$ is a convergent sequence (in $\mathbb{R}$) for every $x \in E$.*

Proof. As we shall see, this is just another form of the *Cantor's Diagonal argument*. Let E be a countable subset of S, say

$$E = \{x_1, x_2, x_3, ...\}$$

Consider the sequence $\{f_n(x_1)\}_{n\in\mathbb{N}}$, which by our hypothesis, is a bounded sequence in $\mathbb{R}$. So, this has a convergent subsequence, say $\{f_{1,k}(x_1)\}_{k\in\mathbb{N}}$ is a convergent subsequence in $\mathbb{R}$. Similarly, consider the sequence $\{f_{1,k}(x_2)\}_{k\in\mathbb{N}}$, which again is a bounded sequence in $\mathbb{R}$. So, there is a convergent subsequence of this, say $\{f_{2,k}(x_2)\}_{k\in\mathbb{N}}$ is a convergent subsequence. Inductively, we can find a convergent subsequence $\{f_{n,k}(x_n)\}_{k\in\mathbb{N}}$ of $\{f_{n-1,k}(x_n)\}$. Writing this in tabular form, we have the following.

$$\begin{aligned} &x_1 : f_{1,1}(x_1)\ ,\ f_{1,2}(x_1)\ ,\ f_{1,3}(x_1)\ ,\ ....\\ &x_2 : f_{2,1}(x_2)\ ,\ f_{2,2}(x_2)\ ,\ f_{2,3}(x_2)\ ,\ ....\\ &\quad \ldots \end{aligned}$$

So, consider the subsequence $\{f_{n,n}\}_{n\in\mathbb{N}}$ of our original sequence $\{f_n\}$. It is clear that $\{f_{n,n}(x_k)\}_{n\in\mathbb{N}}$ converges for every $k \in \mathbb{N}$. This completes the proof. ■

Theorem 1.35 (**Arzela-Ascoli**). *Let S be a compact metric space, and let $\{f_n\}$ be a sequence of real functions on S that is equicontinuous and pointwise bounded at each point of S. Then the following hold.*

(1) *The set $E = \{f_n\}_{n\in\mathbb{N}}$ is uniformly equicontinuous and uniformly bounded on S.*
(2) *$\{f_n\}$ has a uniformly convergent subsequence. In simple words, $\{f_n\}$ has a convergent subsequence in the metric space $(C(S), \rho)$.*

Proof. (1) We first show that E is uniformly bounded. Observe that since S is compact, each f_n is bounded and uniformly continuous on S, and attains its bound. So, for each n, let $x_n \in S$ such that $|f_n(x_n)| = M_n$, where M_n is the upper bound of f_n over S. Let

$$M = \sup_{n\in\mathbb{N}} M_n$$

and we want to show that $M < \infty$. Without loss of generality, suppose $M_n \to M$ as $n \to \infty$ (otherwise we work with a subsequence of M_n converging to M). Also, consider the sequence $\{x_n\}$ in S. Since S is compact, this sequence has a convergent subsequence in S, and again without loss of generality suppose $x_n \to x_0$ for some $x_0 \in S$. Since $\{f_n\}$ is pointwise bounded, there is some M_0 such that

$$|f_n(x_0)| \leq M_0$$

for all $n \in \mathbb{N}$. Moreover, by the equicontinuity of $\{f_n\}$ at x_0, there is some $\delta > 0$ such that for all $y \in B(x_0, \delta)$,

$$|f_n(y) - f_n(x_0)| < 1$$

and this means that

$$|f_n(y)| < |f_n(x_0)| + 1 \leq M_0 + 1$$

for all $y \in B(x_0, \delta)$ and all $n \in \mathbb{N}$. Finally, choose $N \in \mathbb{N}$ such that $d(x_n, x_0) < \delta$ for all $n \geq N$. In that case, we see that

$$|f_n(x_n)| = M_n \leq M_0 + 1$$

for all $n \geq N$, and hence $M < \infty$. This shows that $\{f_n\}$ is uniformly bounded over S.

Next, we show that E is uniformly equicontinuous on S. The proof will be very similar to that of the fact that a continuous function on a compact set is uniformly continuous. Let $x \in S$. Since $\{f_n\}$ is equicontinuous at x, there is some $\delta_x > 0$ such that for all $y \in B(x, \delta_x)$ and all $n \in \mathbb{N}$, we have

$$|f_n(x) - f_n(y)| < \epsilon/2$$

given any $\epsilon > 0$. Consider all balls of the form $B(x, \delta_x/2)$, which form an open cover of S. Since S is compact, this cover admits a finite subcover, say

$$S = B(x_1, \delta_1/2) \cup ... \cup B(x_n, \delta_n/2)$$

where $\delta_i = \delta_{x_i}$ for each i. Put

$$\delta = \text{min}\{\delta_1/2, ..., \delta_n/2\}$$

Now, suppose $x, y \in S$ such that $d(x, y) < \delta$. Now, $x \in B(x_i, \delta_i/2)$ for some i. Clearly, in that case, $y \in B(x_i, \delta_i)$. So, for any $n \in \mathbb{N}$, we have

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_n(y)| < \epsilon$$

which shows that E is uniformly equicontinuous over S. This completes the proof of (1).

(2) Since S is compact, it is separable by **Theorem** 1.2. So, let D be a countable dense subset of S, say

$$D = \{x_1, x_2, x_3, ...\}$$

By **Lemma** 1.34, there is a subsequence of $\{f_{n_k}\}$ that converges at every point of D. Let this subsequence be $\{g_k\}$. We will show that $\{g_k\}$ is the required subsequence by showing that $\{g_k\}$ is uniformly Cauchy, which is just saying that $\{g_k\}$ is a Cauchy sequence in the space $(C(S), \rho)$, and this is enough because by **Theorem** 1.14, $(C(S), \rho)$ is a complete metric space.

Now, by part (1) of the thoerem, we know that the family $\{g_k\}$ is uniformly equicontinuous on S. So, let $\epsilon > 0$ be given, and hence there is some $\delta > 0$ such that

$$d(x, y) < \delta \implies |g_k(x) - g_k(y)| < \epsilon$$

for every $k \in \mathbb{N}$ and $x, y \in S$. So, consider the open balls $B(x_i, \delta)$ for $x_i \in D$. This is an open cover of S, and hence there is some finite subcover, say

$$S = B(x_1, \delta) \cup ... \cup B(x_m, \delta)$$

First, we know that $\{g_k(x_i)\}$ converges for every $1 \leq i \leq m$, and hence is a Cauchy sequence in $\mathbb{R}$. So, choose $N \in \mathbb{N}$ such that $|g_s(x_i) - g_t(x_i)| < \epsilon$ for every $s, t \geq N$ and $1 \leq i \leq m$ (possible because we are only working with finitely many indices i).

Finally, let $x \in S$, so that $x \in B(x_i, \delta)$ for some $1 \leq i \leq m$. Let $s, t \geq N$. So, we have

$$|g_s(x) - g_t(x)| \leq |g_s(x) - g_s(x_i)| + |g_s(x_i) - g_t(x_i)| + |g_t(x_i) - g_t(x)| < 3\epsilon$$

and hence $\{g_k\}$ is uniformly Cauchy, completing the proof of (2).

■

We are moving closer to our original goal of characterising compact subsets of $(C(S), \rho)$. Here is a lemma in going in that direction.

Lemma 1.36. *Let S be a compact metric space, and let $E \subset C(S)$. If E is uniformly bounded and uniformly equicontinuous on S, then so is its closure $\overline{E}$.*

Proof. Suppose E is uniformly bounded and uniformly equicontinuous. First, we show that $\overline{E}$ is uniformly bounded. So, there is some $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in S$ and $f \in E$. Suppose f' is a limit point of E, i.e there is some sequence $\{f_n\}$ in E such that

$$f_n \to f'$$

which equivalently means that f_n converges to f' *uniformly*. So, take $x \in S$. Then, we know that

$$\lim_{n\to\infty} f_n(x) = f'(x)$$

which means

$$\lim_{n\to\infty} |f_n(x)| = |f'(x)|$$

because $|\cdot|$ is a continuous function on $\mathbb{R}$. But, the left hand side in the above limit is $\leq M$, and hence

$$|f'(x)| \leq M$$

showing that $\overline{E}$ is uniformly bounded. Next, we show that $\overline{E}$ is uniformly equicontinuous as well. Again, let f' be a limit point of E, so there is some sequence $\{f_n\}$ converging *uniformly* to f'. Let $\epsilon > 0$ be given. Since $\{f_n\}$ is uniformly equicontinuous, there is some $\delta > 0$ such that

$$d(x, y) < \delta \implies |f_n(x) - f_n(y)| < \epsilon/3$$

for all $x, y \in S$ and $n \in \mathbb{N}$. Now, let $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \epsilon/3$$

for all $x \in S$ (possible by uniform convergence). So, if $d(x, y) < \delta$ for $x, y \in S$, then

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \epsilon$$

and hence this shows that $\overline{E}$ is also uniformly equicontinuous. ■

Finally, we characterise compact subsets of $(C(S), \rho)$. .

Theorem 1.37. *Let S be a compact metric space. Then, a closed subset E of $(C(S), \rho)$ is compact if and only if E is uniformly bounded and uniformly equicontinuous on S. In simpler words, E is compact if and only if it is uniformly equicontinuous on S and is a closed and bounded subset of $(C(S), \rho)$*

Proof. First, suppose E is uniformly bounded and uniformly equicontinuous on S. Then, by the **Arzela-Ascoli Theorem** 1.35, any sequence in E has a convergent subsequence, and since E is closed, this shows that E has the Bolzano-Weierstrass property, and hence E is compact by **Theorem** 1.8. So, we only need to prove the converse.

So suppose E is a compact subset of $(C(S), \rho)$. By part (1) of **Proposition** 1.5, we know that E is a bounded subset of $(C(S), \rho)$. So, by **Proposition** 1.33, we see that E is uniformly bounded over S. Next, we need to show that E is uniformly equicontinuous over S. Since E is compact, it is totally bounded. So, let $\epsilon > 0$ be given. Then, there are finitely many functions $f_1, .., f_m \in E$ such that

$$E \subset B(f_1, \epsilon) \cup ... \cup B(f_m, \epsilon)$$

where the balls are taken in $(C(S), \rho)$. Moreover, since each f_i is continuous over S, it is uniformly continuous. So, there is some $\delta > 0$ such that for any $x, y \in S$ with $d(x, y) < \delta$ and for any $1 \leq i \leq m$,

$$|f_i(x) - f_i(y)| < \epsilon$$

Finally, let $f \in E$ be any function, and let $1 \leq i \leq m$ be such that $f \in B(f_i, \epsilon)$, which means

$$|f_i(x) - f(x)| < \epsilon$$

for all $x \in S$. Let $x, y \in S$ with $d(x, y) < \delta$. Then, we have

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < 3\epsilon$$

and since ϵ was arbitrary, it follows that E is uniformly equicontinuous over S. This completes the proof. ∎

In view of the above theorem and its preceding lemma, we have the following useful corollary.

Corollary 1.37.1. *Let S be compact, and let E be a subset of $(C(S), \rho)$. Then E has compact closure if and only if E is uniformly bounded and uniformly equicontinuous.*

Proof. The forward direction directly follows from **Lemma** 1.36 and **Theorem** 1.37. The backward direction is trivial. ∎

Example 1.8. Consider the space $(C[0,1], \rho)$ with ρ the uniform metric. Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Since $[0,1] \times [0,1]$ is compact, it is clear that K is a bounded function. Put

$$M = \text{sup}\{|K(x,y)| : x, y \in [0,1]\}$$

Next, define

$$(Tf)(x) = \int_0^1 K(x,y) f(y) dy$$

for any $f \in C[0,1]$ and $x \in [0,1]$. It is easy to see that Tf is a continuous function too, and hence $Tf \in C[0,1]$. So, T can be regarded as a map from $C[0,1]$ to $C[0,1]$. $[0,1]$ being compact also means that Tf is a uniformly continuous function on $[0,1]$. (To be completed. See lecture 4 on moodle)

1.13. **Weierstrass Approximation Theorem.** In this section, we will prove an important approximation theorem. We will prove the theorem in three steps. The first two steps will be simple reductions.

Theorem 1.38 (**Weierstrass Approximation Theorem**). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function, where $[a, b]$ is a closed interval. Then, there is a sequence of polynomials $\{P_n\}$ on $[a, b]$ such that $P_n \to f$ uniformly on $[a, b]$, i.e f can be uniformly approximated by polynomials on $[a, b]$.*

Proposition 1.39. *If* **Theorem** 1.38 *holds for a continuous function $f' : [0, 1] \to \mathbb{R}$, then it also holds for a continous function $f : [a, b] \to \mathbb{R}$ where $[a, b]$ is any closed interval.*

Proof. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Consider the map $\gamma : [0, 1] \to [a, b]$ given by

$$\gamma(t) = a + t(b - a)$$

which is a continuous bijection from $[0, 1]$ to $[a, b]$, and infact γ^{-1} is also continuous. Let $f' = f \circ \gamma$, so that $f' : [0, 1] \to \mathbb{R}$ is a continuous function. Let $\{P'_n\}$ be a sequence of polynomials converging uniformly to f' over $[0, 1]$. For $t \in [a, b]$, we define

$$P_n(t) = P'_n(\gamma^{-1}(t)) = P'_n\left(\frac{t - a}{b - a}\right)$$

so in simple terms, $P_n = P'_n \circ \gamma^{-1}$. It is easy to see that each P_n is a polynomial over $[a, b]$. It remains to show that $P_n \to f$ *uniformly* over $[a, b]$. But this is immediate; let $\epsilon > 0$ be given, and so there is some $N \in \mathbb{N}$ such that

$$|P'_n(t) - f'(t)| < \epsilon$$

for all $t \in [0, 1]$ and $n \geq N$. Let $x \in [a, b]$, and let $n \geq N$. So, this means

$$|P'_n(\gamma^{-1}(x)) - f'(\gamma^{-1}(x))| < \epsilon$$

and by the bijectivity of γ and the definition of P_n, we have

$$|P_n(x) - f(x)| < \epsilon$$

which completes the proof. ■

Proposition 1.40. *If* **Theorem** 1.38 *holds for a continuous function $f' : [0, 1] \to \mathbb{R}$ with $f'(0) = f'(1) = 0$, then it holds for any continuous function $f : [0, 1] \to \mathbb{R}$.*

Proof. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Define $f' : [0, 1] \to \mathbb{R}$ by

$$f'(x) = f(x) - f(0) - (f(1) - f(0))x$$

so that $f'(0) = f'(1) = 0$, and f' is continuous. Let $\{P'_n\}$ be a sequence of polynomials converging *uniformly* to f'. Define

$$P_n(x) = P'_n(x) + f(0) + (f(1) - f(0))x$$

and hence P_n is a polynomial over $[0, 1]$ for every $n \in \mathbb{N}$. Moreover, observe that

$$|P'_n(x) - f'(x)| = |P_n(x) - f(x)|$$

for every $n \in \mathbb{N}$ and $x \in [0, 1]$, and it is then clear that $P_n \to f$ *uniformly*, which completes the proof. ■

Proof of **Theorem** 1.38. By the reduction in **Proposition** 1.39 and **Proposition** 1.40, we just need to prove the theorem for a continuous function $f : [0, 1] \to \mathbb{R}$ with $f(0) = f(1) = 0$. (To be completed) ■

Corollary 1.40.1. *Let $[-a,a]$ be a fixed interval. Then, there is a sequence of polynomials $\{P_n\}$ on $[-a,a]$ with $P_n(0)=0$ for every $n\in\mathbb{N}$ such that $P_n\to|\cdot|$ uniformly on $[-a,a]$, where $|\cdot|$ is the absolute value function.*

Proof. By the **Weierstrass Approximation Theorem** 1.38, we know that there is a sequence $\{P_n^*\}$ of polynomials converging uniformly to $|\cdot|$ on $[-a,a]$. In particular, this means that

$$P_n^*(0)\to 0 \text{ as } n\to\infty$$

So, for every $n\in\mathbb{N}$ and $x\in[-a,a]$ define

$$P_n(x)=P_n^*(x)-P_n^*(0)$$

and it is clear that $\{P_n\}$ is the required sequence of polynomials. ∎

Corollary 1.40.2. *The metric space $(C[a,b],\rho)$ is a separable space. Hence, $(C[a,b],\rho)$ is a complete separable metric space where $[a,b]$ is any closed interval.*

Proof. See problem **2.** of **ASSIGNMENT-3**. ∎

1.14. **Convolutions.** In this section, we will define an important operator on the space of functions.

Definition 1.17. Let $f:\mathbb{R}\to\mathbb{R}$ be a function. The *support* of f is defined to be the *closure* of the set

$$C=\{x\in\mathbb{R}|f(x)\neq 0\}$$

If f is continuous and its support is a compact set, then f is said to be a *function of compact support*.

Definition 1.18. Let $f,g\in C(\mathbb{R})$ be continuous functions of compact support. The *convolution* of f and g is defined as

$$f*g(x)=\int_{-\infty}^{\infty}f(y)g(x-y)dy, x\in\mathbb{R}$$

Remark 1.40.1. In the above definition, there is no ambiguity regarding the convergence of the improper integral. Since both functions in the definition have compact support, we can restrict the integral to a closed interval in $\mathbb{R}$.

Let us see two quick properties of the convolution.

Proposition 1.41. *Let f,g be continuous functions with compact support. Then the following hold.*

(1) *$f*g$ is a continuous function.*
(2) *$f*g=g*f$, i.e convolution is a commutative operation.*

Proof. To be completed. ∎

Definition 1.19. Let $\{g_n\}$ be a sequence of continuous functions on $\mathbb{R}$ satisfying the following properties.

(1) $g_n\geq 0$ for $n\geq 1$.
(2) $\int_{-\infty}^{\infty}g_n(x)dx=1$ for $n\geq 1$.
(3) For every $\delta>0$, it is true that

$$\lim_{n\to\infty}\int_{|x|\geq\delta}g_n(x)dx=0$$

Then the sequence $\{g_n\}$ is said to be an *approximate identity*.

Example 1.9. As in the proof of the **Weierstrass Approximation Theorem** 1.38, it can be checked that the sequence $\{\varphi_n\}$ is actually an approximation identity. Moreover, the sequence of polynomials $\{P_n\}$ was precisely the sequence $\{f * \varphi_n\}$, and we showed that this sequence converged uniformly to f. Now, if we define $\{T_n(f)\} = \{f * \varphi_n\}$, then we see that $\rho(T_n(f), f) \to 0$ as $n \to \infty$. So, the sequence $\{T_n\}$ *approximates* the identity map on $C[a,b]$. This justifies the reasoning behind the name of these (and in fact, this convergence is a case of **Theorem** 1.43 below.)

We will now state two theorems without proof, and these will not be used in this course.

Theorem 1.42. *Let f, g be continuous functions with compact support. If $g \in \mathscr{C}^k(\mathbb{R})$, then $f * g \in \mathscr{C}^k(\mathbb{R})$, where $k \in \mathbb{N} \cup \{\infty\}$.*

Remark 1.42.1. A very useful way of looking at this theorem is the *smoothing* property of convolutions, i.e convolutions can be looked at as operators which are used to smoothen a function.

Theorem 1.43. *Let $\{g_n : n \geq 1\}$ be an approximate identity with common compact support, i.e the support of g_n is C for every $n \in \mathbb{N}$, where C is some compact subset of $\mathbb{R}$. Also suppose $g_n \in \mathscr{C}^\infty(\mathbb{R})$ for every $n \in \mathbb{N}$, and let f be a function of compact support. Then $f * g_n \to f$ uniformly on $\mathbb{R}$.*

Remark 1.43.1. This is exactly what was done in the proof of the **Weierstrass Approximation Theorem** 1.38.

Example 1.10. Let

$$\psi(x) = \begin{cases} c \,\text{exp}\left(-\dfrac{1}{(1-|x|)^2}\right) & , |x| < 1 \\ 0 & , |x| \geq 1 \end{cases}$$

where $c > 0$ is chosen so that

$$\int_{-\infty}^{\infty} \psi(x)dx = 1$$

To be completed. See lecture 5 on moodle

1.15. **Towards a generalisation.** We will now try to generalise the approximation theorem we proved in the last section.

Definition 1.20. Let S be a metric space, and let $E \subset S$. Let $\mathcal{E}$ be a family of real valued functions on E such that

(1) $f + g \in \mathcal{E}$,
(2) $fg \in \mathcal{E}$,
(3) $cf \in \mathcal{E}$

for all $f, g \in \mathcal{E}$, i.e $\mathcal{E}$ is closed under addition, multiplication, and scalar multiplications of functions. Then $\mathcal{E}$ is called an *algebra* of functions.

From now on, we let S to be a compact metric space, and we consider the usual space $(C(S), \rho)$ with the uniform metric ρ.

Definition 1.21. A non-empty subset $\mathcal{A}$ of $C(S)$ is called a *subalgebra* of $C(S)$ if $\mathcal{A}$ is an algebra of functions.

Lemma 1.44. *Let $\mathcal{A}$ be a subalgebra of $C(S)$, and let $\mathcal{B}$ be the closure of $\mathcal{A}$ in the uniform metric ρ. Then, $\mathcal{B}$ is also a subalgebra of $C(S)$.*

Proof. This lemma is just saying that sums, products and scalar multiples of uniform limits are also uniform limits. So, let $f, g \in C(S)$ be points of $\mathcal{B}$, and let $\{f_n\}$ and $\{g_n\}$ be sequences of functions in $\mathcal{A}$ such that $f_n \to f$ uniformly and $g_n \to g$ uniformly. Since $\mathcal{A}$ is a subalgebra of $C(S)$, it follows that for any $c \in \mathbb{R}$, $f_n + g_n \in \mathcal{A}$, $f_n g_n \in \mathcal{A}$ and $cf_n \in \mathcal{A}$ for any $n \in \mathbb{N}$. In the following, let $\epsilon > 0$ be fixed.

(1) Let us show that $f_n + g_n \to f + g$ uniformly. We know that there are $N_1, N_2 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon/2$ for all $n \geq N_1, x \in S$ and $|g_n(x) - g(x)| < \epsilon/2$ for all $n \geq N_2, x \in S$. If $N = \max\{N_1, N_2\}$, it immediately follows that
$$|f_n(x) + g_n(x) - f(x) - g(x)| < \epsilon$$
for all $n \geq N$ and $x \in S$, completing the proof. Inparticular, this shows that $f + g \in \mathcal{B}$.

(2) Similarly, we can show that $cf_n \to cf$ uniformly, which will show that $cf \in \mathcal{B}$, and this is very easy and I won't write it here.

(3) Finally, let us show that $f_n g_n \to fg$ uniformly. Consider the identity
$$|f_n g_n - fg| \leq |f_n(g_n - g)| + |g(f_n - f)|$$
Since f_n is a uniformly convergent sequence, it is uniformly bounded, and g is obviously a bounded continuous function. So, it immediately follows that
$$|f_n g_n - fg| \to 0$$
uniformly and hence $f_n g_n \to fg$ uniformly (the arguments here can be made more precise, but this is the general idea).

∎

Remark 1.44.1. Let $\mathcal{A}$ be the subalgebra of all polynomials over the interval $[a, b]$. **Weierstrass Approximation Theorem** 1.38 says that the *uniform closure* (i.e the closure with the uniform metric) of $\mathcal{A}$ in $C[a, b]$ is $C[a, b]$. We will consider a generalisation of this further.

Definition 1.22. Let $\mathcal{A}$ be a subalgebra of $C(S)$. Then $\mathcal{A}$ is said to *separate points* if given any two points $x \neq y \in S$, there is some $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

We now have the generalisation of the approximation theorem.

Theorem 1.45 (Stone-Weierstrass Theorem). *Let (S, d) be a compact metric space, and let $\mathcal{A}$ be a subalgebra of $C(S)$ that separates points and contains the constant functions. Then $\mathcal{A}$ is dense in $(C(S), \rho)$.*

We shall prove this theorem using two important results, which we will prove now. This whole section must be completed! Look at Metric Spaces-6 Notes!

1.16. **Contraction Mapping Theorem.** In this section, we will prove a theorem that is frequently used in a lot of proofs.

Definition 1.23. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a *contraction mapping* if there is some $0 < r < 1$ such that

$$d(Tx, Ty) \leq rd(x, y)$$

for every $x, y \in X$. It is easily seen that a contraction mapping is uniformly continuous.

Theorem 1.46 (**Contraction Mapping Theorem**). *Let (X, d) be a complete metric space, and let T be a contraction mapping on X. Then, T has a unique fixed point, i.e there is a unique $x \in X$ such that $T(x) = x$.*

Proof. Proving uniqueness is almost trivial. Suppose x, y are two points fixed by T. Because T is a contraction mapping, this means

$$d(x, y) \leq rd(x, y)$$

for some $0 < r < 1$, and this is only possible if $x = y$. So, we only need to prove existence of a fixed point.
We will give a constructive proof of the existence of a fixed point, and the idea is actually pretty simple. Let $x_0 \in X$ be a fixed point in X. Define a sequence $\{x_n\}_{n \geq 0}$ by putting

$$x_n = T(x_{n-1})$$

for each $n \in \mathbb{N}$. So, the sequence looks as an infinite iteration

$$\{x_0, T(x_0), T^2(x_0), ...\}$$

Observe that for any $n \geq 1$, we have

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq rd(x_n, x_{n-1})$$

and hence by induction it can be shown that for all $n \geq 1$,

$$d(x_{n+1}, x_n) \leq r^n d(x_1, x_0)$$

Now, we know that the series

$$\sum_{n=0}^{\infty} r^n < \infty$$

So, suppose $\epsilon > 0$ is given. Choose $N \in \mathbb{N}$ such that for all $m > n \geq N$,

$$\sum_{k=n}^{m} r^k < \epsilon$$

Then, suppose $m > n \geq N$. We have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + ... + d(x_{m-1}, x_m) \\ &\leq (r^n + r^{n+1} + ... + r^{m-1}) d(x_1, x_0) \\ &\leq \epsilon d(x_1, x_0) \end{aligned}$$

and hence this shows that the sequence $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in X. So, this sequence is convergent, and let

$$x = \lim_{n \to \infty} x_n$$

Since T is a continuous map, we see that

$$T(x) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

showing that x is a fixed point of T. This completes the proof. ■

Remark 1.46.1. The above theorem need not be true if X is *not* a complete metric space. As a counter example, Complete this

1.17. **ODEs and Picard's Theorem.** In this section, we will see an application of the **Contraction Mapping Theorem** 1.46 to a particular problem involving an *ordinary differential equation* (ODE).

Problem Statement. Let $A > 0$ be a fixed real number, $f : \mathbb{R} \to \mathbb{R}$ a continuous function and $x_0 \in \mathbb{R}$ a given number. We want to find a continuous function $x : [0, A] \to \mathbb{R}$ such that x is differentiable in the open interval $(0, A)$ with

$$x'(t) = f(x(t)) \tag{$*$}$$

for any $t \in (0, A)$, and satisfying the initial condition

$$x(0) = x_0 \tag{$\dagger$}$$

This is a typical example of an *initial value problem*.

An Equivalent Formulation. Consider the following problem: Let A, f and x_0 be as above. We want to find a continuous function $x : [0, A] \to \mathbb{R}$ satisfying the *integral equation*

$$x(t) = x_0 + \int_0^t f(x(s))ds \tag{$\ddagger$}$$

for any $t \in [0, A]$.

Proposition 1.47. *Solving the ODE* $(*)$ *along with the initial value* $(\dagger)$ *is equivalent to solving the integral equation* $(\ddagger)$.

Proof. Suppose x satisfies the ODE $(*)$ along with the initial value $(\dagger)$. Because x and f are assumed to be continuous functions, it follows that $f \circ x$ is continuous as well, and we can extend x' to the interval $[0, A]$ by putting $x'(0) = f(x(0))$ and $x'(A) = f(x(1))$, so that x' is a continuous function on $[0, A]$. By the fundamental theorem of calculus, we then see that for any $t \in [0, A]$,

$$x(t) - x(0) = \int_0^t f(x(s))ds$$

which implies that x satisfies the integral equation $(\ddagger)$. Conversely, suppose x satisfies the integral equation $(\ddagger)$. It is clear that equation $(\dagger)$ is satisfied. By the fundamental of calculus, it easily follows that x is differentiable in the open interval $(0, A)$, and that the derivative of x is

$$x'(t) = f(x(t))$$

for any $t \in (0, A)$, so that $(*)$ is also satisfied. This completes the proof. ■

Theorem 1.48 (Picard's Theorem). *Let* $A > 0$ *and* $x_0 \in \mathbb{R}$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a Lipchitz continuous function, i.e there is some* $K > 0$ *such that*

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in \mathbb{R}$*. Consider the initial value problem given by the equations* $(*)$ *and* $(\dagger)$*. Then there is a unique solution to this initial value problem.*

Proof. The idea of the proof is simple and uses the **Contraction Mapping Theorem** 1.46. By **Proposition** 1.47, it is enough to prove that there is a unique solution to the integral equation in $(\ddagger)$. Let $\alpha > 0$ be any real number such that $K\alpha < 1$. Consider the space $(C[0,\alpha], \rho)$ of continuous functions with the uniform metric ρ. For ease of notation, we will denote the metric ρ by ρ^0. Now, we know that $(C[0,\alpha], \rho_0)$ is a complete metric space. On this space, define a map $T^0 : C[0,\alpha] \to C[0,\alpha]$ by

$$T^0(y)(t) = x_0 + \int_0^t f(y(s))ds \quad , \quad t \in [0,\alpha]$$

for any $y \in C[0,\alpha]$. The fact that T^0 is indeed a map from $C[0,\alpha]$ to itself is a consequence of the fundamental theorem of calculus. Let us show that T^0 is a *contraction mapping* on the complete metric space $(C[0,\alpha], \rho^0)$. To see this, observe that for any $y, z \in C[0,\alpha]$ and $t \in [0,\alpha]$, we have

$$\begin{aligned}
|T^0(y)(t) - T^0(z)(t)| &\leq \int_0^t |f(y(s)) - f(z(s))|ds \\
&\leq K \int_0^t |y(s) - z(s)|ds \\
&\leq K\rho^0(y,z) \int_0^t ds \\
&= K\rho^0(y,z)t \\
&\leq K\alpha\rho^0(y,z)
\end{aligned}$$

Now taking the supremum over all $t \in [0,\alpha]$ in the RHS of the above equation, we get

$$\rho^0(T^0(y), T^0(z)) \leq K\alpha\rho^0(y,z)$$

and since $K\alpha < 1$, this implies that T^0 is a contraction mapping. Hence, by the **Contraction Mapping Theorem** 1.46, T^0 has a unique fixed point in $C[0,\alpha]$. Call this fixed point y^0. So, we see that $y^0 \in C[0,\alpha]$ satisfying

$$y^0(t) = x_0 + \int_0^t f(y^0(s))ds \quad , \quad t \in [0,\alpha]$$

Next, we want to extend y^0 to the interval $[\alpha, 2\alpha]$. This is done in a very similar way. Consider the space $(C[\alpha,2\alpha], \rho^1)$ with the uniform metric ρ^1, and we know that this is a complete metric space. Again, define a mapping $T^1 : C[\alpha,2\alpha] \to C[\alpha,2\alpha]$ by

$$T^1(y)(t) = y^0(\alpha) + \int_\alpha^t f(y(s))ds \quad , \quad t \in [\alpha,2\alpha]$$

for any $y \in C[\alpha,2\alpha]$. By the exact same reasoning as above, it can be shown that T^1 is a *contraction mapping* on this space, and hence there is a unique fixed point $y^1 \in C[\alpha,2\alpha]$ so that

$$y^1(t) = y^0(\alpha) + \int_\alpha^{2\alpha} f(y^1(s))ds \quad , \quad t \in [\alpha,2\alpha]$$

Note that y^1 is an extension of y^0, as $y^0(\alpha) = y^1(\alpha)$, i.e they agree on endpoints. Now we can keep repeating this procedure. The advantage is this: there is some $N \in \mathbb{N}$ such that $(N-1)\alpha < A \le \alpha N$. So, we consider intervals $[0, \alpha], [\alpha, 2\alpha], ..., [(N-1)\alpha, N\alpha]$ and we have functions $y^0, y^1, ..., y^{N-1}$ on each of these N intervals such that

$$y^k((k+1)\alpha) = y^{k+1}((k+1)\alpha)$$

for each $0 \le k \le N-2$, i.e these continuous functions agree on the end points of the corresponding intervals. Moreover, for each $0 \le k \le N-1$ we have

$$y^k(t) = y^{k-1}(k\alpha) + \int_{k\alpha}^{(k+1)\alpha} f(y^k(s))ds \quad , \quad t \in [k\alpha, (k+1)\alpha]$$

where we put $y^{-1}(0) = x_0$.

Now, define the function $x : [0, A] \to \mathbb{R}$ by each of these piecewise functions $y^0, ..., y^{N-1}$. Because these functions are continuous and agree on their end-points, it follows that x is a continuous function. Moreover, it is not hard to see that x satisfies the integral equation

$$x(t) = x_0 + \int_0^t f(x(s))ds \quad , \quad t \in [0, A]$$

and hence there *is* a solution to the given initial value problem. Now, because $y^0, y^1, ..., y^{N-1}$ are *unique*, it follows that x is also unique. This completes the proof. ∎

We now mention a generalised version of this theorem without proof.

Theorem 1.49 (**Generalised Picard's Theorem**). *Let $[A, B]$ be an interval and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipchitz continuous function. Let $t_0 \in (A, B)$ and $x_0 \in \mathbb{R}$. Then there is a unique solution to the initial value problem*

$$x'(t) = f(x(t)) \quad , \quad x \in (A, B)$$
$$x(t_0) = x_0$$

1.18. **Dini's Theorem.** In this section, we will see a theorem about when the limit of a sequence of continuous functions being continuous implies uniform convergence.

Theorem 1.50 (**Dini's Theorem**). *Let (X, d) be a compact metric space, and let $\{f_n\}$ be a sequence of continuous real functions on X such that*

$$f_1 \le f_2 \le f_3 \le ...$$

Also, suppose $f_n \to f$ pointwise on f and assume that f is continuous. Then $\{f_n\}$ converges uniformly to f.

Proof. Let $g_n = f - f_n$, so that g_n is a non-negative continuous function for each $n \in \mathbb{N}$. Let $\epsilon > 0$ be fixed. For any $n \in \mathbb{N}$, put

$$U_n = \{x \in X \mid g_n(x) < \epsilon\}$$

Because each g_n is continuous, U_n is an open set for each $n \in \mathbb{N}$. Moreover, because f_n converges to f pointwise, we see that

$$X = \bigcup_{n \in \mathbb{N}} U_n$$

Because X is compact, there is a finite subcover. So, without loss of generality suppose

$$X = U_1 \cup ... \cup U_N$$

for some $N \in \mathbb{N}$. By the monotonicity assumption, we see that $X = U_N$. But this proves uniform convergence of f_n to f, and this completes the proof. ■

2. Fourier Series

2.1. **Complex Series.**

In this short section, we will mention some facts about series of complex numbers. This will mainly serve as a revision.

Proposition 2.1. *Let $\sum_n a_n$ be a series of complex numbers. If $\sum_n |a_n|$ is convergent, then $\sum_n a_n$ is also convergent. So, absolute convergence implies convergence of a series.*

Theorem 2.2 (**Root Test**). *Let $\sum_n a_n$ be a series of complex numbers, and put*

$$\alpha = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$$

(1) *If $\alpha < 1$, then $\sum_n |a_n| < \infty$.*
(2) *If $\alpha > 1$, then $\sum_n |a_n| = \infty$.*
(3) *If $\alpha = 1$ then nothing can be concluded.*

There is a *root test* as well, which I am not going to include here, as it is a fairly straightforward test.

Definition 2.1. Let $\sum a_n$ and $\sum b_n$ be two series of complex numbers. Define the *convolution* or *product* of these two series as

$$\sum a_n \cdot \sum b_n = \sum c_n$$

where

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

Note that multiplication of two power series is just the convolution of the two series.

Theorem 2.3. *Let $\sum a_n$, $\sum b_n$ be two series of complex numbers. Then, their convolution converges if atleast one of the series is absolutely convergent. Moreover, if both the series are absolutely convergent, then the convolution also converges absolutely.*

The proofs of all these statements can be found in any analysis source like Baby Rudin.

2.2. **Exponential and Trigonometric Functions.**

Here we will review some important functions over the complex numbers.

Definition 2.2. For $z \in \mathbb{C}$, define

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

By the root test, it is clear that all these series converge *absolutely* for every $z \in \mathbb{C}$, and hence the definitions make sense. It is also clear that all these functions are real valued when z is real.

Proposition 2.4 (Euler's Formula). *For any $z \in \mathbb{C}$,*

$$e^{iz} = \cos(z) + i\sin(z)$$

Proof. We have

$$\begin{aligned}
e^{iz} &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{i^n z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{z^{4n}}{(4n)!} + \frac{iz^{4n+1}}{(4n+1)!} - \frac{z^{4n+2}}{(4n+2)!} - \frac{iz^{4n+3}}{(4n+3)!} \\
&= \sum_{n=0}^{\infty} \frac{z^{4n}}{(4n)!} - \frac{z^{4n+2}}{(4n+2)!} + i\sum_{n=0}^{\infty} \frac{z^{4n+1}}{(4n+1)!} - \frac{z^{4n+3}}{(4n+3)!} \\
&= \cos(z) + i\sin(z)
\end{aligned}$$

where we can rearrange the order of summation because of absolute convergence. ∎

Proposition 2.5. *The following properties are true of the exponential function.*

(1) *For any $z, w \in \mathbb{C}$, $e^z e^w = e^{z+w}$.*
(2) *For any $z \in \mathbb{C}$, $e^z e^{-z} = e^0 = 1$. Hence, $e^z \neq 0$ for any $z \in \mathbb{C}$.*
(3) *For any $z \in \mathbb{C}$, $e^{\overline{z}} = \overline{e^z}$.*
(4) *For any $x \in \mathbb{R}$, $|e^{ix}|^2 = 1$.*
(5) *For any $x \in \mathbb{R}$,*

$$\lim_{h \to 0} \frac{e^{i(x+h)} - e^{ix}}{h} = ie^{ix}$$

and this is equivalent to saying that the map $x \mapsto e^{ix}$ has derivative ie^{ix}.

Proof. These properties are more or less a consequence of the absolute convergence of e^z for any z.

(1) By the definition of convolution, we have

$$\begin{aligned} e^z e^w &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{n=0}^{\infty} \frac{w^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!} \\ &= \sum_{k=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (z+w)^n \\ &= e^{z+w} \end{aligned}$$

(2) This easily follows from (1).
(3) Let $z = x + iy$, where $x, y \in \mathbb{R}$. Then, we have

$$\begin{aligned} e^{\overline{z}} &= e^{x-iy} = e^x e^{-iy} = e^x(\cos y - i \sin y) \\ &= \overline{e^x(\cos y + i \sin y)} = \overline{e^x e^{iy}} = \overline{e^z} \end{aligned}$$

where we have used the fact that $\cos x$ is an even function and $\sin x$ is an odd function, which is immediate from the definition.
(4) Let $x \in \mathbb{R}$. Then, we have

$$|e^{ix}|^2 = e^{ix}\overline{e^{ix}} = e^{ix}e^{-ix} = e^0 = 1$$

and hence e^{ix} is on the unit circle for real x.
(5) We have

$$\lim_{h \to 0} \frac{e^{i(x+h)} - e^{ix}}{h} = e^{ix} \lim_{h \to 0} \frac{e^{ih} - 1}{h} = ie^{ix}$$

where the last limit is clear from the power series expansion.

■

We will assume some properties of sin and cos as functions on $\mathbb{R}$, and these can be proven using standard analytical methods.

Proposition 2.6. *The functions* $\sin : \mathbb{R} \to \mathbb{R}$ *and* $\cos : \mathbb{R} \to \mathbb{R}$ *have the following properties.*

(1) *Both are periodic with period 2π, i.e there is a (smallest) positive constant $2\pi \in \mathbb{R}$ such that* $\sin(x + 2\pi) = \sin(x)$ *and* $\cos(x + 2\pi) = \cos(x)$ *for every $x \in \mathbb{R}$.*
(2) *The map $t \to (\cos(t), \sin(t))$ for $t \in [0, 2\pi)$ to the unit circle S^1 is bijective.*

Corollary 2.6.1. *The map $z \to e^z$ is periodic with period $2\pi i$. Moreover, for any $z \in \mathbb{C}$ with $|z| = 1$, there is a unique $\theta \in [0, 2\pi]$ such that $e^{i\theta} = z$.*

Proof. To prove the first claim, we have

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

and the second claim clearly follows from the above proposition. ■

2.3. **More Preliminaries.** (Caution: click here) From this section, the word *integrable* will mean *Lebesgue integrable*, which I don't know yet. However, there is a famous theorem that Riemann Integrable functions are integrable in the Lebesgue sense as well. Moreover, a real/complex valued function f on an interval I is Lebesgue integrable if and only if $|f|$ is Lebesgue integrable. Moreover,

$$\left|\int_I f\right| \leq \int_I |f|$$

Definition 2.3. Let I be an interval in $\mathbb{R}$ (closed, open or half-open). Let $g : I \to \mathbb{C}$ be a complex function. Put

$$g = u + iv$$

so that $g(x) = u(x) + iv(x)$ for $x \in I$. We say that g is *integrable* on I if both u, v are integrable on I. In such a case, define

$$\int_I g(x)dx = \int_I u(x)dx + i\int_I v(x)dx$$

Proposition 2.7. *Let $g_1, g_2 : I \to \mathbb{C}$ be integrable. Then, for any $c_1, c_2 \in \mathbb{C}$,*

$$\int_I c_1g_1 + c_2g_2 = c_1\int_I g_1 + c_2\int_I g_2$$

Proof. Immediate. ■

Definition 2.4. Let I be any interval in $\mathbb{R}$. Define

$$L^2(I) := \left\{f : I \to \mathbb{C} \ : \ \int_I |f(x)|^2 < \infty\right\}$$

On this space, define the *inner product* of two functions $f, g \in L^2(I)$ as

$$\langle f, g\rangle = \int_I f(x)\overline{g(x)}dx$$

and this definition of the inner product is very similar to the usual Hermitian product on $\mathbb{C}^n$. Finally, for any $f \in L^2(I)$, define the *norm* of f by

$$||f|| = \sqrt{\langle f, f\rangle}$$

and it is immediate that $||f|| \geq 0$.

Proposition 2.8. *Consider the space $L^2(I)$. Put*

$$d(f, g) = ||f - g||$$

for any $f, g \in L^2(I)$. Then $(L^2(I), d)$ is a pseudo-metric space.

Proof. See **ASSIGNMENT-1** of Calculus, specifically for the proof of the triangle inequality. ■

Remark 2.8.1. As we've seen before, the reason why this is a pseudo-metric space and not just a metric space is because there non-zero functions whose integral over I is zero.

Proposition 2.9. *Let I be a closed and bounded interval in $\mathbb{R}$, and let f be a continuous complex function on I. Show that $f \in L^2(I)$.*

Proof. This is immediate from the fact that continuous functions over a closed and bounded interval in $\mathbb{R}$ are Riemann integrable. ■

Scratch this section. I won't delete this section from the pdf, but the prof. decided not to go into measure theory. So the section 2.3. **More Preliminaries.** will be present, but can be discarded without any hesitation. We will deal only with complex continuous functions of a real variable, which will make life easy as continuous functions are already Riemann integrable.

2.4. **The Objective of Fourier Series.** Let f be a continuous complex valued periodic function on $\mathbb{R}$ with period 2π. The question is whether we can write it in the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $c_n \in \mathbb{C}$ and $n \in \mathbb{Z}$. Notice that we are summing on $\mathbb{Z}$, and since $\mathbb{Z}$ is countable, this is a usual infinite series. Moreover, there will be no ambiguity in the rearrangement of this series, as the series will turn out to be absolutely convergent. A series of the above form is called a *Fourier Series* and the coefficients c_n are called the *Fourier Coefficients* of f.

Definition 2.5. Consider the space

$$H := C([-\pi, \pi], \mathbb{C})$$

i.e the space of complex continuous functions on $[-\pi, \pi]$ with the uniform metric ρ. By **Theorem** 1.14, we know that this is a complete metric space (the same proof works for complex functions as well). We will only be interested in functions in H with $f(-\pi) = f(\pi)$ (i.e period 2π) and we will use the notation $I = [-\pi, \pi]$.

Definition 2.6. Let $f \in H$, so we can write

$$f = u + iv$$

so that $f(x) = u(x) + iv(x)$ for every $x \in I$. Define

$$\int_I f := \int_I u + i \int_I v$$

and the definition makes sense since both u, v are continuous.

Proposition 2.10. *For any* $f \in H$,

$$\int_I |f|^2 < \infty$$

Proof. This is clear since f is a continuous function. ■

Definition 2.7. On H, define the *inner product*

$$\langle f, g \rangle := \int_I f(x)\overline{g(x)}dx$$

for any $f, g \in H$, and define the *norm* as

$$||f|| := \sqrt{\langle f, f \rangle}$$

and it is clear that $||f|| \geq 0$ for every $f \in H$.

Remark 2.10.1. Since we are dealing with *continuous functions*, it turns out that the given norm makes H into a normed linear space. As we show below, the given inner product is a *Hermitian product*. So, this inner product is a positive definite Hermitian product over H.

Lemma 2.11. *Let $f, g, h \in H$ and $c \in \mathbb{C}$. Then the following hold.*

(1) $\langle f, g\rangle = \overline{\langle g, f\rangle}$.
(2) $\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$.
(3) $\langle cf, g\rangle = c\langle f, g\rangle$.
(4) $||cf|| = |c|||f||$.
(5) *(Cauchy-Schwarz inequality).* $|\langle f, g\rangle| \leq ||f||||g||$.
(6) $||f+g|| \leq ||f|| + ||g||$.
(7) *Putting* $d(f, g) = ||f-g||$ *makes* H *into a metric space.*

Proof. For sake of simplicity, we will write $f = f_1 + if_2$ for any $f \in H$.

(1) We see that for any continuous complex function F,
$$\int_I \overline{F} = \overline{\int_I F}$$
which easily follows from the definition of the integral. So, we have
$$\langle f, g\rangle = \int_I f(x)\overline{g(x)}dx = \overline{\int_I \overline{f(x)}g(x)dx} = \overline{\int_I g(x)\overline{f(x)}dx} = \overline{\langle g, f\rangle}$$
(2) This property easily follows from the linearity of the integral.
(3) This also follows from the linearity of the integral.
(4) This follows from the definition of the norm.
(5) This property is true for any vector space with a positive definite Hermitian product on it. See the beginning section of the **Analysis-2** notes for this.
(6) This easily follows follows from (4), because:
$$\begin{aligned} ||f+g||^2 &= \langle f+g, f+g\rangle \\ &= \langle f, f\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle \\ &\leq ||f||^2 + 2|\langle f, g\rangle| + ||g||^2 \\ &\leq ||f||^2 + 2||f||||g|| + ||g||^2 \\ &= (||f|| + ||g||)^2 \end{aligned}$$
and this completes the proof by taking square roots.
(7) This is immediate, because H contains only *continuous functions*.

■

Definition 2.8. Let $S = \{\varphi_0, \varphi_1, \varphi_2, ...\}$ be a collection of elements of H. If
$$\langle \varphi_m, \varphi_n\rangle = 0$$
for $m \neq n$, then S is said to be an *orthogonal system* in H. If, in addition
$$||\varphi_n|| = 1$$
for all $n = 0, 1, 2, ..$ then S is called an *orthonormal system* in H.

Example 2.1. As before, let $I = [-\pi, \pi]$. Put
$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$$
for any $x \in I$ and $n \in \{0, \pm 1, \pm 2, \pm 3...\}$. We see that
$$\frac{1}{2\pi}\int_I e^{imx}dx = \begin{cases} 1 & , m = 0 \\ 0 & , m \neq 0 \end{cases}$$

So in this case, $\varphi_n, n \in \{0, \pm 1, \pm 2, \pm 3, ...\}$ is an orthonormal system in H.

Lemma 2.12. *Let the sequence* $\{f_n\}$ *in* H *converge uniformly to* $f \in H$*, i.e* $\rho(f_n, f) \to 0$*. Then,*

$$||f_n - f|| \to 0$$

and for any $g \in H$*,*

$$\langle f_n, g\rangle \to \langle f, g\rangle$$

Moreover, we have that

$$||f|| = \lim_{n\to\infty} ||f_n||$$

Proof. Suppose $\rho(f_n, f) \to 0$. Observe that

$$||f_n - f||^2 = \int_I |f_n - f|^2 < 2\pi\epsilon^2$$

because for eventually large n, we know that $|f_n - f| < \epsilon$ over the interval I, and since ϵ is arbitrary this means that

$$||f_n - f|| \to 0$$

To prove the second claim, let $g \in H$. Observe that

$$|\langle f_n, g\rangle - \langle f, g\rangle| = |\langle f_n - f, g\rangle| \leq ||f_n - f|| \cdot ||g||$$

where we applied the Cauchy-Schwarz inequality in the last step. So, it follows that

$$\langle f_n, g\rangle \to \langle f, g\rangle$$

Finally, it is straightforward to check that

$$|\,||f_n|| - ||f||\,| \leq ||f_n - f||$$

(this is just a consequence of the properties of norms), and hence by the first claim it follows that

$$||f_n|| \to ||f||$$

and this completes the proof. ■

2.5. **Some Important Computations.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is a complex continuous periodic function with period 2π. Also, suppose f has a representation

$$f(x) = \sum_{n=\infty}^{\infty} c_n e^{inx} \tag{•}$$

in the interval $I = [-\pi, \pi]$. For every $N \in \mathbb{N}$, define

$$s_N(x) = \sum_{n=-N}^{N} c_n e^{inx} \tag{••}$$

and suppose the sequence $\{s_N\}$ of functions on I converges *uniformly* to f on I, i.e

$$s_N \to f \text{ (uniformly)} \tag{$\star$}$$

(and this will be our assumption throughout this section).

Proposition 2.13. *If* f *is as above, then for any* $k \in \mathbb{Z}$*,*

$$c_k = \frac{1}{2\pi}\int_I f(x)e^{-ikx}dx \tag{†}$$

Proof. By **Lemma** 2.12, we see that for $k \in \mathbb{Z}$,

$$\langle s_N, e^{ikx} \rangle \to \langle f, e^{ikx} \rangle$$

Writing this explicitly, we see that

$$\int_I f(x)e^{-ikx}dx = \lim_{N\to\infty} \int_I s_N(x)e^{-ikx}dx \tag{$*$}$$

It is clear that each $s_N(x)$ is a continuous function on I, and can be easily integrated. So, we see that

$$\begin{aligned}
\lim_{N\to\infty} \int_I s_N(x)e^{-ikx}dx &= \lim_{N\to\infty} \int_I \sum_{n=-N}^{N} c_n e^{inx}e^{-ikx}dx \\
&= \lim_{N\to\infty} \sum_{n=-N}^{N} c_n \int_I e^{inx-ikx}dx \\
&= \lim_{N\to\infty} \sum_{n=-N}^{N} c_n \langle e^{inx}, e^{ikx} \rangle \\
&= \lim_{N\to\infty} 2\pi c_k \\
&= c_k 2\pi
\end{aligned}$$

where in the second last step we used the results in **Example** 2.1 (in particular, we used the orthogonality of the functions occurring in the inner product). Combining equation $(*)$ with the above result, we see that

$$c_k = \frac{1}{2\pi} \int_I f(x)e^{-ikx}dx$$

for any $k \in \mathbb{Z}$. This completes the proof. ■

Proposition 2.14. *For any $N \in \mathbb{N}$, let s_N be as in equation (●●). Then,*

$$||s_N||^2 = 2\pi \sum_{n=-N}^{N} |c_n|^2$$

Proof. This is just a straightforward computation as below.

$$
\begin{aligned}
||s_N||^2 &= \int_I |s_N(x)|^2 dx \\
&= \int_I s_N(x)\overline{s_N(x)} dx \\
&= \int_I \left(\sum_{n=-N}^{N} c_n e^{inx}\right)\left(\sum_{n=-N}^{N} \overline{c_n} e^{-inx}\right) dx \\
&= \int_I \sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N} c_{n_1}\overline{c_{n_2}} e^{in_1x}e^{-in_2x} dx \\
&= \sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N} c_{n_1}\overline{c_{n_2}} \int_I e^{in_1x}e^{-in_2x} dx \\
&= \sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N} c_{n_1}\overline{c_{n_2}} \langle e^{in_1x}, e^{in_2x}\rangle \\
&= \sum_{n_1=-N}^{N} |c_{n_1}|^2 2\pi \\
&= 2\pi \sum_{n=-N}^{N} |c_n|^2
\end{aligned}
$$

where we have again used the orthogonality relations in **Example** 2.1. This completes the proof. ■

Theorem 2.15 (**Minimality of Fourier coefficients**). *Let $f \in H$, and let $N > 0$ be a fixed positive integer. Among all choice of $\{d_i : -N \le i \le N\}$ the expression*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(x) - \sum_{n=-N}^{N} d_n e^{inx}\right|^2 dx$$

is uniquely minimised when $d_n = c_n$ for all $|n| \le N$, where c_n is defined as in equation (†) in **Proposition** 2.13*. Moreover, the minimum value is*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx - \sum_{n=-N}^{N}|c_n|^2$$

Proof. Since $f \in H$, i.e f is a continuous function, the coefficients c_n are well defined for $|n| \le N$. Let $d_n \in \mathbb{C}$ be chosen for every $|n| \le N$, and suppose

$$d_n = c_n + \alpha_n$$

for some $\alpha_n \in \mathbb{C}$. We have the following equalities.

$$
\begin{aligned}
(f(x)\overline{d_n e^{inx}}) + (\overline{f(x)} d_n e^{inx}) &= 2\text{Re}(f(x)\overline{d_n e^{inx}}) \\
&= 2\text{Re}(\overline{d_n} f(x) e^{-inx}) \qquad (\star\star)
\end{aligned}
$$

for each $|n| \le N$ and $x \in I$. Now, as in **Example** 2.1, we know that

$$\left\{\frac{1}{\sqrt{2\pi}} e^{ikx} : k \in \mathbb{Z}\right\}$$

is an orthonormal system, and hence we get the following chain of equalities.

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(x)-\sum_{n=-N}^{N}d_ne^{inx}\right|^2dx\\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(x)-\sum_{n=-N}^{N}d_ne^{inx}\right)\left(\overline{f(x)}-\sum_{n=-N}^{N}\overline{d_ne^{inx}}\right)dx\\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx-\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-N}^{N}f(x)\overline{d_ne^{inx}}+\overline{f(x)}d_ne^{inx}\right)dx+\\
&\quad\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m,n=-N}^{N}d_m\overline{d_n}e^{imx-inx}\right)dx\\
&=\frac{1}{2\pi}||f||^2-\sum_{n=-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}2\text{Re}(\overline{d_n}f(x)e^{-inx})dx+\frac{1}{2\pi}\sum_{m,n=-N}^{N}d_m\overline{d_n}\langle e^{imx},e^{inx}\rangle\\
&=\frac{1}{2\pi}||f||^2-\sum_{n=-N}^{N}\frac{1}{2\pi}2\text{Re}\int_{-\pi}^{\pi}\overline{d_n}f(x)e^{-inx}dx+\sum_{n=-N}^{N}|d_n|^2\\
&=\frac{1}{2\pi}||f||^2-2\text{Re}\sum_{n=-N}^{N}\overline{d_n}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx+\sum_{n=-N}^{N}|d_n|^2\\
&=\frac{1}{2\pi}||f||^2-2\text{Re}\sum_{n=-N}^{N}c_n\overline{d_n}+\sum_{n=-N}^{N}|d_n|^2
\end{aligned}$$

From the last equation, we use the fact that $d_n = c_n + \alpha_n$, and we get the following (for easy understanding, I will rewrite the last equation again).

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(x)-\sum_{n=-N}^{N}d_ne^{inx}\right|^2dx\\
&=\frac{1}{2\pi}||f||^2-2\text{Re}\sum_{n=-N}^{N}c_n\overline{d_n}+\sum_{n=-N}^{N}|d_n|^2\\
&=\frac{1}{2\pi}||f||^2-2\text{Re}\sum_{n=-N}^{N}(|c_n|^2+c_n\overline{\alpha_n})+\sum_{n=-N}^{N}(|c_n|^2+c_n\overline{\alpha_n}+\overline{c_n}\alpha_n+|\alpha_n|^2)\\
&=\frac{1}{2\pi}||f||^2-2\sum_{n=-N}^{N}(|c_n|^2+\text{Re }c_n\overline{\alpha_n})+\sum_{n=-N}^{N}(|c_n|^2+2\text{Re }c_n\overline{\alpha_n}+|\alpha_n|^2)\\
&=\frac{1}{2\pi}||f||^2-\sum_{n=-N}^{N}|c_n|^2+\sum_{n=-N}^{N}|\alpha_n|^2
\end{aligned}$$

and hence the claim follows from the last equality. ■

Corollary 2.15.1 (**Bessel's Inequality**). *Let $f \in H$ and let c_n be as defined in* (†) *in* **Proposition** 2.13*. Then,*

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

Also,

$$\frac{1}{2\pi} ||s_N||^2 \leq \frac{1}{2\pi} ||f||^2$$

for any $N \in \mathbb{N}$.

Proof. This follows immediately from **Theorem** 2.15. In particular, we see that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$$

and hence this series converges absolutely. The second claim follows immediately from **Proposition** 2.14. ■

Corollary 2.15.2 (**Riemann-Lebesgue Lemma**). *Let $f \in H$ and let c_n be as defined in* (†) *in* **Proposition** 2.13*. Then*

$$\lim_{|n| \to \infty} c_n = 0$$

Moreover, we have

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = 0$$

Proof. The first claim is immediate from the convergence of

$$\sum_{n \in \mathbb{Z}} |c_n|^2$$

For the second claim, let $n > 0$. Then observe that

$$\lim_{n \to \infty} c_n = \lim_{n \to \infty} c_{-n} = 0$$

which means

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) e^{inx} dx = 0$$

So, we get that

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x)[e^{inx} + e^{-inx}] dx = 2 \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = 0$$

and similarly

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x)[e^{inx} - e^{-inx}] dx = 2 \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = 0$$

and this completes the proof of the second claim. ■

2.6. **Convergence Results.** Having carried out important computations in the last section, we will now move to the question of convergence.

Theorem 2.16. *Let $f \in H$ such that $f \in \mathscr{C}^1(\mathbb{R})$ and f is periodic with period 2π. Let s_N be as defined in (••) where c_n is defined as in equation (†) in* **Proposition** *2.13. Then $\{s_N\}$ converges uniformly.*

Proof. By the hypothesis it is clear that f and f' are continuous periodic functions on $\mathbb{R}$ with period 2π. Let $\{d_n\}$ be the Fourier coefficients of f', i.e

$$d_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx$$

for any $n \in \mathbb{Z}$. Using integration by parts, we see that

$$\begin{aligned} d_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\left[f(x)e^{-inx}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)(-in)e^{-inx}dx\right] \\ &= \frac{1}{2\pi}in\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ &= inc_n \end{aligned}$$

for any $n \in \mathbb{Z}$. So we see that

$$\begin{aligned} \sum_{n\in\mathbb{Z}-\{0\}} |c_n| &= \sum_{n\in\mathbb{Z}-\{0\}} \frac{|d_n|}{|n|} \\ &\leq \left(\sum_{n\in\mathbb{Z}-\{0\}} |d_n|^2\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}-\{0\}} \frac{1}{|n|^2}\right)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{2\pi}}\sqrt{\int_{-\pi}^{\pi} |f'(x)|^2 dx}\left(\sum_{n\in\mathbb{Z}-\{0\}} \frac{1}{|n|^2}\right)^{\frac{1}{2}} \\ &< \infty \end{aligned}$$

where above we have used the Cauchy-Schwarz inequality in the second step, **Bessel's Inequality** 2.15.1 for d_n in the third step and the fact that $\sum_{n=0}^{\infty}\frac{1}{n^2}$ is a convergent series. So, it follows that

$$\sum_{n\in\mathbb{Z}} |c_n| < \infty$$

and hence this series converges. Now observe that for any $n \in \mathbb{Z}$,

$$|c_n e^{inx}| = |c_n|$$

and hence by the Weierstrass M-test, it follows that the sequence $\{s_N\}$ converges uniformly. ■

Remark 2.16.1. As you may have observed, we are using non-traditional notations for writing countable infinite sums. However, this leads to no ambiguity, because all series we are considering converge absolutely, so we can work with any rearrangement of the series.

Definition 2.9. For any $N \geq 0$, put

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} \ , \ x \in \mathbb{R}$$

The sequence of functions $\{D_N\}$ is called the *Dirichlet Kernel*.

Lemma 2.17. *Each D_N is a continuous periodic function with period 2π. Also,*

$$D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin \frac{1}{2}x} \ , \ x \in \mathbb{R} - \{0\}$$

Moreover,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1$$

Proof. That each D_N is continuous and periodic with period 2π is clear. Moreover, $D_N(x)$ is a geometric series with common ratio e^{ix} for every $x \in \mathbb{R}$. So using the geometric sum formula, we see that for $x \neq 0$

$$\begin{aligned} D_N(x) &= (e^{-iNx})\frac{e^{i(2N+1)x} - 1}{e^{ix} - 1} \\ &= \frac{e^{i(N+1)x} - e^{-iNx}}{e^{ix} - 1} \\ &= \frac{(e^{i(N+1)x} - e^{-iNx})e^{-\frac{1}{2}ix}}{(e^{ix} - 1)e^{-\frac{1}{2}ix}} \\ &= \frac{(e^{iNx+\frac{1}{2}ix} - e^{-iNx-\frac{1}{2}ix})}{e^{\frac{1}{2}ix} - e^{-\frac{1}{2}ix}} \\ &= \frac{\sin((N+\frac{1}{2})x)}{\sin\frac{1}{2}x} \end{aligned}$$

The last assertion is immediate by the results in **Example** 2.1. ■

Lemma 2.18. *Let $f \in H$ be a continuous periodic function on $\mathbb{R}$ with period 2π. Let $N > 0$ be any integer, and let D_N be as defined in* **Definition** 2.9, *and let s_N be as defined in* (••). *Then*

$$s_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt$$

Proof. By the definitions of s_N, D_N and c_n, we see that

$$\begin{aligned}
s_N(x) &= \sum_{n=-N}^{N} c_n e^{inx} \\
&= \sum_{n=-N}^{N} \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(t)e^{-int}dt\right) e^{inx} \\
&= \sum_{n=-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{in(x-t)}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{n=-N}^{N} e^{in(x-t)}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)dt \\
&= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-u)D_N(u)du \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)D_N(u)du
\end{aligned}$$

In the second last step, we have used a change of variables. In the last step, by the periodicity of f, it doesn't matter which interval we integrate on as long as its length is 2π. ■

We now prove the existence of a uniformly convergent Fourier Series for $\mathscr{C}^1$ functions.

Theorem 2.19. *Let $f \in \mathscr{C}^1(R)$ be a periodic function with period 2π. Let s_N be defined as in (••). Then $\{s_N\}$ converges to f uniformly.*

Proof. By **Theorem** 2.16, we have already shown that $\{s_N\}$ converges uniformly. So, it is enough to show that s_N converges to f *pointwise*.

Let $x \in \mathbb{R}$ be fixed. By **Lemma** 2.17 and **Lemma** 2.18, we have the following chain of equalities for any $N \geq 0$

$$\begin{aligned}
s_N(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)D_N(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x-t)-f(x))D_N(t)dt
\end{aligned}$$

Now we define a function $g : [-\pi, \pi] \to \mathbb{R}$ by

$$g(t) = \begin{cases} \dfrac{f(x-t)-f(x)}{\sin \frac{1}{2}t} & , t \neq 0 \\ -2f'(x) & , t = 0 \end{cases}$$

So it follows that g is a continuous function on I, i.e $g \in H$ (the way we have defined g at 0 makes it continuous). Also, for any $t \in [-\pi, \pi]$ we can write

$$(f(x-t)-f(x))D_N(t) = g(t)\sin\left(\left(N+\frac{1}{2}\right)t\right)$$

and here we are using **Lemma** 2.17. So, we see that

$$s_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\sin\left(\left(N+\frac{1}{2}\right)t\right)dt$$

$$(**) \qquad = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(g(t)\sin\left(\frac{1}{2}t\right)\right)\cos(Nt)dt + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(g(t)\cos\left(\frac{1}{2}t\right)\right)\sin(Nt)dt$$

Now observe that the functions $t \mapsto g(t)\sin\left(\frac{1}{2}t\right)$ and $t \mapsto g(t)\cos\left(\frac{1}{2}t\right)$ are both elements of H. So by The **Riemann-Lebesgue Lemma** 2.15.2, we see that both the integrals in equation $(**)$ go to 0 as $N \to \infty$. This completes the proof. ■

Corollary 2.19.1 (Parseval's Identity). *Let $f \in \mathscr{C}^1(R)$ be periodic with period 2π. Then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \frac{1}{2\pi}||f|| = \sum_{n=-\infty}^{\infty}|c_n|^2$$

where c_n are the Fourier coefficients of F.

Proof. By **Theorem** 2.19, we see that $\{s_N\}$ converges to f uniformly on $[-\pi,\pi]$. This means that $\{|s_N|\}$ converges to $|f|$ uniformly, and since we are dealing with bounded functions, this means that $\{|s_N|^2\}$ converges to $|f|^2$ uniformly. So we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|^2dt = \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}|s_N(t)|^2dt = \lim_{N\to\infty}\frac{1}{2\pi}||s_N||^2$$

By **Proposition** 2.14, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|^2dt = \sum_{n=-\infty}^{\infty}|c_n|^2$$

completing the proof. ■

2.7. Fejer's Theorem.

Note that **Theorem** 2.19 refers only to $\mathscr{C}^1$ functions. We will now state some facts without proof which will generalise the situation to continuous periodic functions.

Definition 2.10. Let $\{a_n\}_{n\geq 0}$ be a sequence of complex numbers. Put

$$s_n = \sum_{i=0}^{n} a_i$$

so $\{s_n\}_{n\geq 0}$ is the sequence of partial sums. Denote

$$\sigma_n = \frac{s_0 + s_1 + \ldots + s_n}{n+1}$$

and so $\{\sigma_n\}$ is the sequence of arithmetic means of s_n. If $\{\sigma_n\}$ converges, then the sequence $\{a_n\}$ is said to be *Cesaro Summable*, and the limit $\sigma = \lim \sigma_n$ is called the *Cesaro sum* of $\{a_n\}$.

Proposition 2.20. *If a series $\sum_{n\geq 0} a_n$ is convergent with value s, then the sequence $\{a_n\}$ is Cesaro summable and it's Cesaro sum is s.*

Proposition 2.21. *Suppose the sequence $\{a_n\}$ is Cesaro summable with Cesaro sum σ. Also suppose that the sequence $\{na_n\}$ is bounded. Then the series $\sum_{n\geq 0} a_n$ is convergent with sum σ.*

Theorem 2.22 (**Fejer's Theorem**). *Let f be a continuous periodic function on $\mathbb{R}$ with period 2π. Let s_N be defined as in (••). Let $\{\sigma_N\}$ be the sequence of arithmetic means of $\{s_N\}$. Then $\sigma_N \to f$ uniformly over $[-\pi, \pi]$, and hence on all of $\mathbb{R}$ by the periodicity of f and σ_N.*