COMPLEX ANALYSIS

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These are solutions to some problems in Complex Analysis. These were in conjunction with the instructor notes.

Contents

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1. Lecture 1

1.1. **Proof of Proposition 1.1.** In this exercise, we will show that any open subset of $\mathbb{C}$ is connected if and only if it is path connected. So, let $A \subseteq \mathbb{C}$ be any open subset, and first suppose that A is path connected. We claim that A is connected; for the sake of contradiction, suppose A is *not* connected, that is $A = U \sqcup V$ where U, V are non-empty disjoint open subsets of A. Let $x \in U$ and $y \in V$, and let $f : [0, 1] \to A$ be a path in A from x to y. Consider the open subsets $f^{-1}(U)$ and $f^{-1}(V)$ of $[0, 1]$. Clearly, these intervals are disjoint and their union is $[0, 1]$. But, this contradicts the connectedness of $[0, 1]$. So, it follows that A is indeed connected (note that in any path-connected topological space is connected by the same argument).

Conversely, suppose A is open and connected. Recall that in any connected space, the only sets which are both open and closed are ϕ and A. So, let $a \in A$ be any point. Let A_1 be the set of all points of A which are connected via paths to a, and let A_1' be its complement in A. Clearly, $A_1 \neq \phi$ (because $a \in A_1$). We will show that A_1 is both open and closed, and it will follow that $A_1 = A$, which will show that A is path-connected. It is easy to show that A_1 is open; take any point $x \in A_1$, i.e a is connected to x by a path. Since A is open, there is a ball centered at x which is completely contained inside A. Clearly, any point x' in this ball can be connected to a via a path, by concatenating the path from a to x and the linear path from x to x'. So, A_1 is an open set. To show that A_1 is closed, we can very similarly show that A_1' is open. This completes the proof.

1.2. **Exercise (1).** The proof of the fact that any connected open subset of $\mathbb{R}^n$ is path-connected is given in section 1.1 above. We will show that the *topologists sine curve*, i.e the closure of

$$\left\{\left(x, \sin\frac{1}{x}\right) \mid x \in (0,1)\right\}$$

inside $\mathbb{R}^2$ is connected but not path-connected. Observe that the above set is the image of $(0,1)$ under the continuous map

$$x \mapsto \left(x, \sin\frac{1}{x}\right)$$

and hence this set is connected. Because closures of connected sets are connected, it follows that the *topologists sine curve* is also connected. Now, we will show that this curve is *not* path-connected. Need to complete this!

1.3. **Exercise (2).** Let n be any positive integer. We will show that $\mathbb{R}^n$ is complete. To show this, use two facts.

(1) The norms $||\cdot||_\infty$ and $||\cdot||_2$ on $\mathbb{R}^n$ are equivalent (or just use the more stronger fact that all norms in $\mathbb{R}^n$ are equivalent).
(2) Now just use the fact that $\mathbb{R}$ is complete.

1.4. **Exercise (3).** This problem is trivial once we know the fact that the map $t \mapsto (\cos t, \sin t)$ is bijective from $[0, 2\pi)$ onto the unit circle S^1.

1.5. **Exercise (4).** Consider the equation $z^n = \overline{z}^n$. If $z = r(\cos\theta + i\sin\theta)$, this equation in polar coordinates corresponds to

$$r^n(\cos n\theta + \sin n\theta) = r^n(\cos n\theta - \sin n\theta)$$

Solving this, we get

$$2i\sin n\theta = 0$$

and from here the values of θ can be computed.

2. Lecture 2

2.1. **Simpler proof of Theorem 2.8.** *Let $U \subseteq \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be a function. Let $c = a + bi \in U$, and write $f = u + iv$, where u and v are real functions on U. Then f is differentiable at c if and only if u, v are differentiable at (a, b) (as functions from $U \to \mathbb{R}$) and their partial derivatives satisfy the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x}(a,b) = \frac{\partial v}{\partial y}(a,b) \text{ and } \frac{\partial u}{\partial y}(a,b) = -\frac{\partial v}{\partial x}(a,b)$$

Further, when this happens, it is true that

$$f'(c) = \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b) = \frac{\partial v}{\partial y}(a,b) - i\frac{\partial u}{\partial y}(a,b)$$

Proof. First, suppose f is (complex) differentiable at c. To show that u, v are differentiable as functions from $U \to \mathbb{R}$, it is enough to show that f is real differentiable as a function $U \to \mathbb{R}^2$. First, let $h \in \mathbb{R}$. Then, we have the following chain of equations.

$$\begin{aligned} f'(c) &= \lim_{h\to 0} \frac{f(h+c)-f(c)}{h} \\ &= \lim_{h\to 0} \frac{u(h+c)+iv(h+c)-u(c)-iv(c)}{h} \end{aligned}$$

From here, separate the real and imaginary parts to get the following.

$$\begin{aligned} f'(c) &= \lim_{h\to 0} \frac{u(h+c)-u(c)}{h} + i\lim_{h\to 0} \frac{v(h+c)-v(c)}{h} \\ &= \lim_{h\to 0} \frac{u(a+h,b)-u(a,b)}{h} + i\lim_{h\to 0} \frac{v(a+h,b)-v(a,b)}{h} \end{aligned}$$

and this implies that both $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ exist at the point (a, b), and we immediately see that

$$f'(c) = \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b)$$

Next, we can replace h by ih in the above equations, and we will obtain that both $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ exist at the point (a, b), and that

$$f'(c) = \frac{\partial v}{\partial y}(a,b) - i\frac{\partial u}{\partial y}(a,b)$$

From the last two equations, we obtain the Cauchy-Riemann equations. It remains to show that f is real differentiable at (a, b). But we have the derivative with us! It is simply

$$f'(a,b) = \begin{bmatrix} \frac{\partial u}{\partial x}(a,b) & \frac{\partial u}{\partial y}(a,b) \\ \frac{\partial v}{\partial x}(a,b) & \frac{\partial v}{\partial y}(a,b) \end{bmatrix} = \lambda$$

Checking this is a simple calculation. Suppose $h = (h_1, h_2) \in \mathbb{R}^2$, or we can write $h = h_1 + ih_2 \in \mathbb{C}$. Then using the Cauchy-Riemann equations, we have

$$\begin{aligned} \lim_{h\to 0} \frac{||f(c+h)-f(c)-\lambda(h)||}{||h||} &= \lim_{h\to 0} \left| \frac{f(c+h)-f(h)-\lambda(h)}{h} \right| \\ &= \lim_{h\to 0} \left| \frac{f(c+h)-f(c)-\left(\frac{\partial u}{\partial x}(a,b)h_1 + \frac{\partial u}{\partial y}(a,b)h_2, \frac{\partial v}{\partial x}(a,b)h_1 + \frac{\partial v}{\partial y}(a,b)h_2\right)}{h} \right| \\ &= \lim_{h\to 0} \left| \frac{f(c+h)-f(c)-\left(\frac{\partial u}{\partial x}(a,b)h_1 - \frac{\partial v}{\partial x}(a,b)h_2, \frac{\partial v}{\partial x}(a,b)h_1 + \frac{\partial u}{\partial x}(a,b)h_2\right)}{h} \right| \\ &= \lim_{h\to 0} \left| \frac{f(c+h)-f(c)-(h_1+ih_2)\left(\frac{\partial u}{\partial x}(a,b) + i\frac{\partial u}{\partial v}(a,b)\right)}{h} \right| \\ &= \lim_{h\to 0} \left| \frac{f(c+h)-f(c)-hf'(c)}{h} \right| \\ &= 0 \end{aligned}$$

and this shows that f is real differentiable at (a, b), and so are u, v.

Conversely, if u, v are real differentiable at (a, b), then so is f. In addition, suppose u, v satisfy the Cauchy-Riemann equations. So, all that needs to be shown is that f is complex differentiable at c. But this can easily be checked as above by putting

$$f'(c) = \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b)$$

This completes the proof. ∎

2.2. **Exercise (1).** (Very Boring) Here, we will prove the rules of differentiation for complex functions. So let $c \in \mathbb{C}$, and f, g functions defined in a neighborhood of c and differentiable at c, h a function defined on a neighborhood of $f(c)$ and differentiable at $f(c)$, and $\alpha \in \mathbb{C}$. Then we have the following.

(1) $(f + \alpha g)'(c) = f'(c) + \alpha g'(c)$.

Proof. This is an easy computation. We have the following chain of equations.

$$\begin{aligned}(f + \alpha g')(c) &= \lim_{h\to 0} \frac{(f+\alpha g)(c+h) - (f+\alpha g)(c)}{h} \\ &= \lim_{h\to 0} \frac{f(c+h) - f(c) + \alpha g(c+h) - \alpha g(c)}{h} \\ &= f'(c) + \alpha g'(c)\end{aligned}$$

∎

(2) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

Proof. This one has a neat trick. We do it as follows.

$$\begin{aligned}(fg)'(c) &= \lim_{h\to 0} \frac{(fg)(c+h) - (fg)(c)}{h} \\ &= \lim_{h\to 0} \frac{f(c+h)g(c+h) - f(c)g(c)}{h} \\ &= \lim_{h\to 0} \frac{f(c+h)g(c+h) + f(c)g(c+h) - f(c)g(c+h) - f(c)g(c)}{h} \\ &= \lim_{h\to 0} \frac{[f(c+h) - f(c)]g(c+h)}{h} + \lim_{h\to 0} \frac{[g(c+h) - g(c)]f(c)}{h} \\ &= f'(c)g(c) + f(c)g'(c)\end{aligned}$$

∎

(3) $(h \circ f)'(c) = h'(f(c))f'(c)$. To be completed.

(4) $\left(\frac{1}{f}\right)'(c) = -\frac{f(c)}{f(c)^2}$ if $f(c_1) \neq 0$ for every c_1 in a neighborhood of c. To be completed.

2.3. **Exercise (2).** Let

$$f(z) = \begin{cases} z^5|z|^{-4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

We now determine $\mathfrak{R}(f)$ and $\mathfrak{I}(f)$ as functions of real variables x and y, where $z = x + iy$. The following computation is straightforward.

$$\begin{aligned}
\frac{z^5}{|z|^4} &= \frac{(x+iy)^5}{(x^2+y^2)^2} \\
&= x^5 + \binom{5}{1}x^4(iy) + \binom{5}{2}x^3(iy)^2 + \binom{5}{3}x^2(iy)^3 + \binom{5}{4}x(iy)^4 + \binom{5}{5}(iy)^5 \\
&= \frac{x^5 + 5x^4(iy) + 10x^3(iy)^2 + 10x^2(iy)^3 + 5x(iy)^4 + (iy)^5}{(x^2+y^2)^2} \\
&= \frac{x^5 - 10x^3y^2 + 5xy^4}{(x^2+y^2)^2} + i\frac{(5x^4y - 10x^2y^3 + y^5)}{(x^2+y^2)^2}
\end{aligned}$$

and hence we see that

$$\mathfrak{R}(f)(x,y) = \frac{x^5 - 10x^3y^2 + 5xy^4}{(x^2+y^2)^2} \quad , \quad \mathfrak{I}(f)(x,y) = \frac{5x^4y - 10x^2y^3 + y^5}{(x^2+y^2)^2}$$

Next, we show that $\mathfrak{R}(f)$ and $\mathfrak{I}(f)$ satisfy the Cauchy-Riemann equations at $z = 0$. But this is clear by the following.

$$\begin{aligned}
\frac{\partial \mathfrak{R}(f)(0,0)}{\partial x} &= 1 \\
\frac{\partial \mathfrak{R}(f)(0,0)}{\partial y} &= 0 \\
\frac{\partial \mathfrak{I}(f)(0,0)}{\partial x} &= 0 \\
\frac{\partial \mathfrak{I}(f)(0,0)}{\partial y} &= 1
\end{aligned}$$

Finally, we show that $f'(0)$ does not exist. Let $r \in \mathbb{R}$, and first let $h = r$. Then, we see that

$$\lim_{h\to 0}\frac{f(h)}{h} = \lim_{r\to 0}\frac{f(r)}{r} = \lim_{r\to 0}\frac{r}{r} = 1$$

Next, put $h = (1+i)r$. So we have

$$\lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\frac{f((1+i)r)}{(1+i)r} = \lim_{r\to 0}\frac{-4r^5 - i4r^5}{4|r|^4 r(1+i)} = \lim_{r\to 0}\frac{-r^4}{|r^4|} = -1$$

and hence it follows that $f'(0)$ does not exist.

2.4. **Exercise (3).** We define

$$\begin{aligned}
f_z &= \frac{1}{2}(f_x - if_y) \\
f_{\overline{z}} &= \frac{1}{2}(f_x + if_y)
\end{aligned}$$

(a) Let us treat z and $\overline{z}$ as the independent coordinates. Then, our composite mapping is

$$(z, \overline{z}) \mapsto \left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) \mapsto f\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right)$$

So, using the chain rule of differentiation, we see that

$$\begin{bmatrix} f_z & f_{\overline{z}} \end{bmatrix} = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & \frac{-1}{2i} \end{bmatrix}$$

and hence the above equation agrees with our definitions.

(b) If f is differentiable at a point c, then we know from **Theorem** 2.8 that $f'(c) = u_x(c) + iv_x(c)$. Now, from the Cauchy-Riemann equations, we see that

$$u_x(c) + iv_x(c) = \frac{1}{2}\left[u_x(c) + v_y(c) + i(v_x(c) - u_y(c))\right] = \frac{1}{2}\left[f_x(c) - if_y(c)\right] = f_z(c)$$

and hence $f'(c) = f_z(c)$. Similarly, the Cauchy Riemann equations will simplify to give $f_{\overline{z}}(c) = 0$.

2.5. **Exercise (4).** Let $f = z^m \overline{z}^n$ with $m, n \geq 0$. We show that

$$f_z = mz^{m-1}\overline{z}^n \quad , \quad f_{\overline{z}} = nz^m\overline{z}^{n-1}$$

Complete this using polar coordinates

2.6. **Exercise (5).** Consider the function

$$f(x+yi) = \begin{cases} \dfrac{xy^2(x+yi)}{x^2+y^4} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$$

We show that f is not differentiable at the origin. Writing $f = u + vi$, we see that

$$u(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^4} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$$

and that

$$v(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^4} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$$

For f to be differentiable at 0, both u, v must be differentiable at $(0,0)$ and they must satisfy the Cauchy-Riemann equations. Observe that

$$\frac{\partial u}{\partial x}(0,0) = \frac{\partial v}{\partial x}(0,0) = \frac{\partial u}{\partial y}(0,0) = \frac{\partial v}{\partial y}(0,0) = 0$$

and so if u, v are differentiable at 0, their derivatives must be 0. This implies that

$$\lim_{(x,y)\to(0,0)} \frac{|v(x,y)|}{\sqrt{x^2+y^2}} = 0$$

Now, consider the curve $x = y^2$. Taking the above limit over this curve, we see that

$$\lim_{y\to 0} \frac{|y^5|}{2y^4\sqrt{y^4+y^2}} = \lim_{y\to 0} \frac{|y^5|}{2|y^5|\sqrt{y^2+1}} \neq 0$$

and this is a contradiction. So, it follows that v is not differentiable, i.e f is not complex differentiable at 0.

2.7. **Exercise (6).** Let $f(z)$ be a function defined in a neighborhood of $c \in \mathbb{C}$. We show that $f(z)$ is differentiable at c if and only if $\overline{f(\overline{z})}$ is differentiable at $\overline{c}$.

First, suppose $f(z)$ is differentiable at the point c. Clearly, we see that $\overline{f(\overline{z})}$ is defined in a neighborhood of $\overline{c}$. Put $g(z) = \overline{f(\overline{z})}$. This is to say that $g(z)$ is defined in a neighborhood of $\overline{c}$. We claim that

$$g'(\overline{c}) = \overline{f'(c)}$$

This is just a straightforward computation. We have the following.

$$\begin{aligned}
\lim_{h\to 0} \frac{g(\overline{c}+h) - g(\overline{c})}{h} &= \lim_{h\to 0} \frac{\overline{f(\overline{\overline{c}+h})} - \overline{f(c)}}{h} \\
&= \lim_{h\to 0} \frac{\overline{f(c+\overline{h})} - \overline{f(c)}}{h} \\
&= \lim_{h\to 0} \frac{\overline{f(c+\overline{h}) - f(c)}}{h} \\
&= \lim_{h\to 0} \overline{\left(\frac{f(c+\overline{h}) - f(c)}{\overline{h}}\right)} \\
&= \overline{f'(c)}
\end{aligned}$$

and in the last step above, we have interchanged the conjugation operator with the limit, because conjugation is continuous. The converse is similarly proven.

2.8. **Exercise (7).** Here we will find the Cauchy-Riemann equations in polar coordinates. If f is some complex function, we write

$$f = u + iv$$

We define a map g by $g = f \circ \gamma$, where $\gamma : \mathbb{C} \to \mathbb{C}$ is the map

$$\gamma(r, \theta) = (r\cos\theta, r\sin\theta)$$

By the chain rule, we see that

$$\begin{bmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

So, this corresponds to the four equations

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
\frac{\partial u}{\partial \theta} &= -\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta \\
\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \\
\frac{\partial v}{\partial \theta} &= -\frac{\partial v}{\partial x} r\sin\theta + \frac{\partial v}{\partial y} r\cos\theta
\end{aligned}$$

and so the Cauchy-Riemann equations in polar coordinates are

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad , \quad r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}$$

3. Lecture 3

3.1. Exercise (6). Here we will look at two important limits.

(1) The first limit is $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. For many amazing proofs, check out this link.
(2) The second limit is $\lim_{n\to\infty} \binom{n}{k}^{\frac{1}{n}} = 1$ for every $k \in \mathbb{N}$. Observe that

$$\binom{n}{k}^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}(n-1)^{\frac{1}{n}}...(n-(k-1))^{\frac{1}{n}}}{(k!)^{\frac{1}{n}}}$$

Using the limit in (1), and the fact that $\sqrt[n]{k!} \to 1$ as $n \to \infty$, the claim follows.

3.2. Exercise (8). It is clear that $\mathbb{C}\{z\}$ is a subring of $\mathbb{C}[[z]]$, because sums and products of convergent power series are themselves convergent power series. We will now show that if $a_0 \neq 0$, then the inverse of $\sum a_n z^n$ in $\mathbb{C}[[z]]$ belongs to $\mathbb{C}\{z\}$. To be completed.

4. Lecture 4

4.1. Fact used in Proposition 4.4. In the proof of **Proposition 4.4** of the notes, the following fact has been used: if

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}$$

is an absolutely convergent double series of complex numbers, then

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_{ji}$$

4.2. Exercise (1). Let U be a domain, $c_0 \in \mathbb{C}$ and $\tau : U \to \mathbb{C}$ be the map $c \mapsto c + c_0$. Clearly, τ is a continuous map, begin a translation. It also has a two sided inverse, namely the map τ^{-1} given by $c \mapsto c - c_0$, and being a translation τ^{-1} is also a continuous map. Clearly, τ is a homeomorphism; hence, $\mathrm{Im}(\tau)$ is an open connected set, i.e $\mathrm{Im}(\tau)$ is a domain in $\mathbb{C}$.

Suppose $x_0 \in U$ is a point such that a map $f : U \to \mathbb{C}$ is differentiable at x_0. We show that $f \circ \tau^{-1}$ is differentiable at the point $\tau(x) \in \tau(U)$. But this is obvious by the chain rule, since τ^{-1} is differentiable at each point in $\tau(U)$.

Next, suppose $f : U \to \mathbb{C}$ is analytic on U. We show that $f \circ \tau^{-1}$ is analytic on $\mathrm{Im}(\tau)$. To show this, let $c \in \mathrm{Im}(\tau)$. Consider the point $\tau^{-1}(c)$. We know that there is some $\epsilon > 0$ such that $B_{\tau^{-1}(c),\epsilon} \subseteq U$ and

$$(*) \qquad f(z) = \sum_{n\in\mathbb{N}} a_n(z - \tau^{-1}(c))^n$$

for all $z \in B_{\tau^{-1}(c),\epsilon}$. Observe that

$$\tau(B_{\tau^{-1}(c),\epsilon}) = B(c,\epsilon) \subseteq \mathrm{Im}(\tau)$$

So, if $z \in B(c,\epsilon)$, then $(*)$ implies that

$$f \circ \tau^{-1}(z) = \sum_{n\in\mathbb{N}} a_n(\tau^{-1}(z) - \tau^{-1}(c))^n = \sum_{n\in\mathbb{N}} a_n(z-c)^n$$

and this proves that $f \circ \tau^{-1}$ is analytic on $\mathrm{Im}(\tau)$.

4.3. **Exercise (2).** We can also prove the same when τ is just a scaling map, i.e $\zeta \mapsto c_0\zeta$ for some $c_0 \neq 0$. The proof is easy. The statement about $f \circ \tau$ being holomorphic is again follows by the chain rule. So, we will only prove the statement about analyticity.

So, suppose $f : U \to \mathbb{C}$ is analytic on U. Let $c \in \text{Im}(\tau)$. Consider the point $\tau^{-1}(c) \in U$. We know that there is some $\epsilon > 0$ such that $B_{\tau^{-1}(c),\epsilon} \subseteq U$ and that for any $z \in B_{\tau^{-1}(c),\epsilon}$

$$f(z) = \sum_{n\in\mathbb{N}} a_n(z - \tau^{-1}(c))^n \tag{†}$$

Observe that

$$\tau(B_{\tau^{-1}(c),\epsilon}) = B(c, c_0\epsilon) \subseteq \text{Im}(\tau)$$

So, if $z \in B(c, c_0\epsilon)$, then (†) implies that

$$f \circ \tau^{-1}(z) = \sum_{n\in\mathbb{N}} a_n(\tau^{-1}(z) - \tau^{-1}(c))^n = \sum_{n\in\mathbb{N}} \frac{a_n}{c_0}(z - c)^n$$

and hence it follows that $f \circ \tau^{-1}$ is analytic on $\text{Im}(\tau)$.

4.4. **Exercise (3).** Let $f : U \to \mathbb{C}$ be an analytic function on a domain U such that $f^{(k)}(z) = 0$ for every $z \in U$. Let $c \in U$ be any point. We know that locally around c, f can be written as a power series, that is

$$f(z) = \sum_{n\in\mathbb{N}} a_n(z - c)^n$$

for all z in a ball centered at c. Now, restrict f to this ball. We know that f is infinitely differentiable, and the coefficients a_k are given by

$$a_m = m!f^{(m)}(c)$$

Since $f^k = 0$, it follows that f is a polynomial of degree atmost $k - 1$ in this ball. Let this polynomial be p. So, we have shown that $f \equiv p$ on a non-empty open set contained in U. So, it follows that $f - p$ is identically zero on a non-empty open set contained in U. However, by **Proposition 4.6** in the main notes, we know that the zeroes of $f - p$ in U will be isolated if $f - p$ is not identically zero, since $f - p$ is analytic on U. So, it follows that $f - p$ is identically zero on U, i.e f is a polynomial of degree atmost $k - 1$. This completes our proof.

4.5. **Exercise (4).** This is an alternative proof of the fact that the zeroes of an analytic function that is not identically zero are isolated.

4.6. **Exercise (5).** To be completed.

5. Lecture 5

5.1. **Exercise (1).** Points number (1) and (2), i.e the power series expansions of sin and cos are clear by the power series expansion of e^z.

The formula $e^{iz} = \cos z + i \sin z$ is also clear from the definition of these functions.

The rest of the properties are also easy to see from the definitions.

5.2. **Exercise (4).** Here we expand $1/z$ as a power series around $z = 1$, and we compute its radius of convergence. So suppose

$$\frac{1}{z} = \sum_{n\in\mathbb{N}} a_n(z-1)^n$$

in some neighborhood of 1. We see that

$$a_n = \left(\frac{1}{z}\right)^{(n)}(1)\frac{1}{n!} = (-1)^n$$

and hence we see that

$$\frac{1}{z} = \sum_{n\in\mathbb{N}}(-1)^n(z-1)^n$$

Clearly, the radius of convergence of this series is 1.

5.3. **Exercise (6).** Let U be a domain not containing 0 and let f, g be branches of the logarithm on U. Consider the function

$$h(z) := \frac{f(z) - g(z)}{2\pi i}$$

on U. We have

$$e^{2\pi i h(z)} = e^{f(z)-g(z)} = 1$$

because f, g are branches of the logarithm. This means that $h(z) \in \mathbb{Z}$ for all $z \in U$. Since h is continuous, it follows that $h(z) = n$ for some $n \in \mathbb{Z}$. This implies that

$$f(z) - g(z) = 2n\pi i$$

i.e any two branches of the logarithm differ by an interger multiple of $2\pi i$. Conversely, if $f(z) - g(z) = 2\pi i$, it is clear that f is a branch of a logarithm if and only if g is.

5.4. **Exercise (7).** Let U be a domain not containing 0 and let f be a branch of the logarithm on U. We show that f is holomorphic on U.

First, suppose $c \in U \setminus (-\infty, 0]$. Take a ball $B_{c,R}$ which does not intersect $(-\infty, 0]$; on this neighborhood, $f(z)$ differs from $\mathrm{Log}(z)$ by a holomorphic function (in particular, any two branches of the logarithm differ by an integer multiple of $2\pi i$), and hence $f(z)$ is also holomorphic at c. Moreover, it is clear that $f'(z) = 1/z$ on $U \setminus (-\infty, 0]$.

Next, suppose $c \in U \cap (-\infty, 0]$. The idea is to *rotate the domain* where $\mathrm{Log}(z)$ is differentiable by an appropriate angle θ. For any angle θ, note that the function

$$\mathrm{Log}(e^{i\theta}z) - i\theta$$

is holomorphic on $\mathbb{C} \setminus K$, where K is the negative x-axis $(-\infty, 0]$ rotated counter-clockwise by the angle θ. So, let $\theta \neq 2k\pi$ by any angle. Choose $B_{\epsilon,c}$ such that $B_{c,\epsilon} \subseteq U \setminus K$ (possible because U is open). Then, again, $f(z)$ and $\mathrm{Log}(e^{i\theta}z) - i\theta$ differ by a multiple of $2\pi i$, and hence f is differentiable at c. Again, we have

$$f'(c) = (\mathrm{Log}(e^{i\theta}z) - i\theta)'_{z=c} = \frac{1}{z}$$

and this proves the claim.

6. Lecture 6

6.1. **Exercise (1).** Here we show that $\int_\gamma f(z)dz$ is independent of the choice of the partition. But this is really easy, and I'll just give a proof sketch. Suppose

$$P := a = t_0 < t_1 < ... < t_n = b$$

is a partition of $[a, b]$. Now let P' be any *refinement* of this partition. We show that the integral $\int_\gamma f(z)dz$ taken with respect to P is equal to the one taken with respect to the refinement P'. But this is trivial, by the additivity of integrals. Then, given two partitions P_1 and P_2 of $[a, b]$, the claim follows by taking their common refinement and using **Lemma 6.4**.

6.2. **Exercise 3.** Let U be a domain. We will show that between any two points of U, there is a piecewise-linear path between them.

Let $a_0 \in U$ be any point. We show that the set

$$S := \{a \in U \mid \text{There is a piecewise-linear path between } a \text{ and } a_0\}$$

Clearly, S is non-empty, because $a_0 \in S$. We show that S is both open and closed in U, and this will show that $S = U$, which will complete our proof.

Suppose $a \in S$. Take a ball $B_{a,\epsilon}$ such that $B_{a,\epsilon} \subseteq U$ (possible because U is open). Since $a \in S$, there is a piecewise-linear path from a_0 to a. If $\zeta \in B_{a,\epsilon}$, we can extend this piecewise-linear path to a path from $a_0 \to \zeta$, by simply concatenating the line segment from a to ζ to our path. This shows that $\zeta \in S$, i.e S is open.

To show that S is closed, let $p \in U$ be a limit point of S. Again, take a ball $B_{p,\epsilon}$ such that $B_{p,\epsilon} \subseteq U$. Now, $S \cap B_{p,\epsilon} \neq \phi$ since p is a limit point of S. Again, by concatening a piecewise-linear path with a straight line, we otain that $p \in S$, showing that S is closed in U. This completes our proof.

7. Lecture 7

7.1. **Exercise (1).** Here we show that γ_2 is *not* a reparametrization of γ, where γ, γ_2 are as in **Example 7.2**. As mentioned in the example, the intuitive idea is that γ goes around the circle *once*, while γ_2 revolves around the circle *twice*. We use this idea to distinguish between the two maps.

For the sake of contradiction, suppose γ_2 is a reparametrisation of γ, i.e

$$\gamma_2 = \gamma \circ \tau$$

where $\tau : [0, 2] \to [0, 1]$ is a non-decreasing piecewise differentiable surjective map. It is easy to see that $\gamma_2^{-1}(1, 0) = \{0, 1, 2\}$. We will show that $(\gamma \circ \tau)^{-1}(1, 0)$ cannot be this set, and that will be our contradiction. Note that

$$(\gamma \circ \tau)^{-1}(1, 0) = \tau^{-1}(\gamma^{-1}(1, 0))$$

Now we know that $\gamma^{-1}(1, 0) = \{0, 1\}$. Because τ is a non-decreasing surjective map, it is clear that $\tau(0) = 0$ and $\tau(2) = 1$, i.e $\{0, 2\} \subseteq \tau^{-1}(\{0, 1\})$. Now, we also have that $\tau(1) \in \{0, 1\}$. If $\tau(1) = 0$, then it follows that $\tau[0, 1] = 0$, which contradicts the fact that $\tau^{-1}\{0, 1\} = \{0, 1, 2\}$. Similarly, a contradiction is obtained if $\tau(1) = 1$. This proves the claim.

8. Lecture 8

8.1. Exercise (1). Let $f : [a,b] \to \mathbb{C}$ be a function. We show that

$$\left|\int_a^b f\mathrm{d}t\right| \leq \int_a^b |f|\mathrm{d}t$$

To see this, let $\theta = \arg \int_a^b f\mathrm{d}t$. Observe that

$$e^{-i\theta}\int_a^b f\mathrm{d}t = \int_a^b e^{-i\theta} f\mathrm{d}t$$

an this implies that

$$\mathrm{Re}\left[e^{-i\theta}\int_a^b f\mathrm{d}t\right] = \int_a^b \mathrm{Re}[e^{-i\theta}f]\mathrm{d}t \leq \int_a^b |f|\mathrm{d}t$$

Also, the extreme left hand side in the above equation is just

$$\mathrm{Re}\left[e^{-i\theta}\int_a^b f\mathrm{d}t\right] = \left|\int_a^b f\mathrm{d}t\right|$$

and this completes the proof.

8.2. Exercise (2). Here we complete the assertion (2) $\iff$ (3) of **Proposition 8.4**. Throughout this proof, U is a domain and $f : U \to \mathbb{C}$ is a continuous function.

First, suppose there is some function $F : U \to \mathbb{C}$ such that for all piecewise-differentiable paths $\gamma : [a,b] \to U$ it is true that

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

If γ is a piecewise-differentiable closed path in U, the right hand side in the above equality becomes 0, and we get that

$$\int_\gamma f(z)\mathrm{d}z = 0$$

which proves the forward direction.

Conversely, suppose for every piecewise-differentiable closed path γ in U, it is true that

$$\int_\gamma f(z)\mathrm{d}z = 0$$

Fix a point $a_0 \in U$. Now, for any point $a \in U$, we know that there is a piecewise-differentiable path γ_a in U from a_0 to a. Define the map $F : U \to \mathbb{C}$ by

$$F(a) = \int_{\gamma_a} f(z)\mathrm{d}z$$

First, we need to show that F is well-defined, i.e the choice of the path γ_a does not matter. γ_a and γ_a' are two piecewise-differentiable paths from a_0 to a. So, observe that the path $\gamma_a - \gamma_a'$ is a path from a_0 to itself. By our hypothesis, we know that

$$\int_{\gamma_a - \gamma_a'} f(z)\mathrm{d}z = \int_{\gamma_a} f(z)\mathrm{d}z - \int_{\gamma_a'} f(z)\mathrm{d}z = 0$$

and this implies that

$$\int_{\gamma_a} f(z)\mathrm{d}z = \int_{\gamma_a'} f(z)\mathrm{d}z$$

and this proves the well-definedness of F. Next, we claim that if $\gamma : [a, b] \to U$ is any piecewise-differntiable path in U then

$$\int_\gamma f(z)\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

To see this, let γ_a and γ_b be piecewise-differentiable paths in U from a_0 to $\gamma(a)$ and a_0 to $\gamma(b)$ respectively. Then, observe that

$$\gamma_a + \gamma - \gamma_b$$

is a path from a_0 to a_0. So, we see that

$$\int_{\gamma_a+\gamma-\gamma_b} f(z)\mathrm{d}z = \int_{\gamma_a} f(z)\mathrm{d}z + \int_\gamma f(z)\mathrm{d}z - \int_{\gamma_b} f(z)\mathrm{d}z = 0$$

and this implies that

$$\int_\gamma f(z)\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

and this completes the proof.

8.3. **Exercise (3).** Let r be a positive real number, and let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the path given by $t \mapsto re^{it}$. We show that

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = 2\pi i$$

But this is clear, because

$$\int_\gamma \frac{1}{z}\mathrm{d}z = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it}dt = 2\pi i$$

Now, let γ be a piecewise-differentiable closed path that avoids some ray in $\mathbb{C}$ (by a ray we mean the set $\{re^{i\alpha} \mid r \in \mathbb{R}\ , r \geq 0\}$ for some α). Then

$$\int_\gamma \frac{1}{z}\mathrm{d}z = 0$$

This is because the function $1/z$ has a primitive on $\mathbb{C} - \{re^{i\alpha} \mid r \in \mathbb{R}\ , r \geq 0\}$ obtained by rotating the domain where $\mathrm{Log}(z)$ is differentiable, just like we did in **Exercise (7)** of Lecture 5.

9. Lecture 10

9.1. **Exercise (1).** In this exercise, we show that it can be assumed that U is centered at 0, where U is as in the statement of **Theorem 10.1**. So, let c be the centre of U. Let $\tau : U \to \mathbb{C}$ be the map $z \mapsto z - c$, and let $U_1 = \mathrm{Im}(\tau)$. Then it is clear that τ is a homeomorphism between U and U_1. Let $f_1 = f \circ \tau^{-1}$ and let $\gamma_1 = \tau \circ \gamma$. Then we will show that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma_1} f_1(z)\,\mathrm{d}z$$

Suppose the path γ is defined on the interval $[a, b]$. Let $a = t_0 < t_1 < ... < t_k = b$ be a good partition for γ. It is clear that this is a good partition for γ_1 as well, because

τ is a holomorphic function on $\mathbb{C}$, and hence we can use the chain rule. Now, observe that

$$\begin{aligned}\int_{\gamma_1} f_1(z)\,\mathrm{d}z &= \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} f_1(\gamma_1(t))\gamma_1'(t)\,\mathrm{d}t\\ &= \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} f(\tau^{-1}(\gamma_1(t)))\tau'(\gamma(t))\gamma'(t)\,\mathrm{d}t\\ &= \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} f(\gamma(t))\gamma'(t)\,\mathrm{d}t\\ &= \int_\gamma f(z)\,\mathrm{d}z\end{aligned}$$

and this completes the proof.

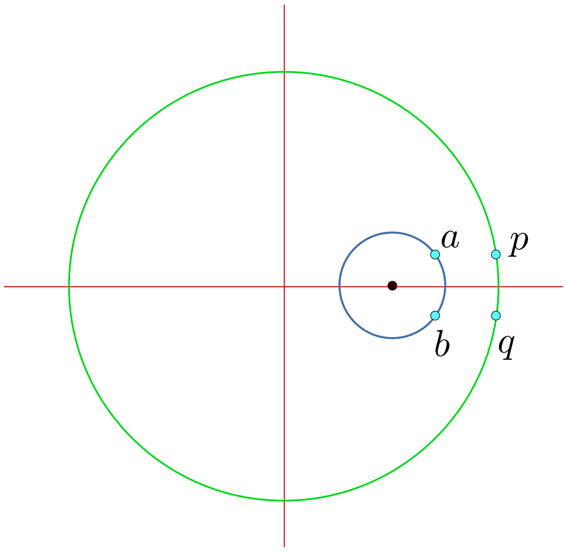
9.2. **Exercise (3).** Let U be a domain containing $\overline{B_{0,1}}$ and let $\gamma : [0,1] \to \mathbb{C}$ be the path given by $t \mapsto e^{2\pi i t}$. In this exercise, we will compute

$$\int_\gamma \frac{1}{z-\frac{1}{2}}\,\mathrm{d}z$$

(a) Let $0 < r \ll 1$ and $\sigma : [0,1] \to \mathbb{C}$ be the closed path $t \mapsto \frac{1}{2} + re^{2\pi i t}$. Since $r \ll 1$, we can assume that $B_{\frac{1}{2},r} \subseteq \overline{B_{0,1}}$. Now, we have

$$\begin{aligned}\int_\sigma \frac{1}{z-\frac{1}{2}}\,\mathrm{d}z &= \int_0^1 \frac{1}{\frac{1}{2}+r\,e^{2\pi i t}-\frac{1}{2}}\sigma'(t)\,\mathrm{d}t\\ &= \int_0^1 \frac{1}{re^{2\pi i t}} 2\pi i r\, e^{2\pi i t}\,\mathrm{d}t\\ &= 2\pi i\end{aligned}$$

(b) Let $0 < \epsilon \ll 1$. Consider the following picture.



The path γ is the circle in green; the path σ is the smaller circle in blue; the points p, q, a and b are as marked in the figure. The path γ_1 is the path from p to q following the green circle counter clockwise. The path σ_1 is the path from a to b following the blue circle counter-clockwise. τ_1 and τ_2 are the paths from a to p and b to q respectively, which are both parallel to the real axis.

Let Γ be the piecewise-differentiable closed path given by

$$\Gamma = \gamma_1 - \tau_2 - \sigma_1 + \tau_1$$

We show that

$$\int_\Gamma \frac{1}{z - \frac{1}{2}} \,\mathrm{d}z = 0 \tag{9.1}$$

Observe that $\Gamma \subseteq U \setminus \left\{\frac{1}{2} + r \mid r \in \mathbb{R}, r \geq 0\right\}$, and we know that there is a branch of the logarithm on an open subset of $\mathbb{C}$ minus a ray in $\mathbb{C}$, as in **Exercise (7)** of Lecture 5. So, it follows that the integrand in (9.1) has a primitive on $U \setminus \left\{\frac{1}{2} + r \mid r \in \mathbb{R}, r \geq 0\right\}$, and hence the given integral is zero.

(c) This part is really easy. Will complete it soon.

(d) This follows by taking limits as $\epsilon \to 0$ in equation (9.1).

(e) Note that there is nothing special about the point $\frac{1}{2}$ here; we can repeat the same exact argument for any other point in $B_{0,1}$ by taking a suitable disk around the point that is contained in $B_{0,1}$ and then define similar paths as above.

10. Lecture 13

10.1. **Exercise (1).** We have to show that $f(U) \cap (-\infty, 0] = \phi$. The function we have is $f : \mathbb{C} - \{\zeta'\} \to \mathbb{C}$ given by

$$f(z) = \frac{z - \zeta}{z - \zeta'}$$

and U is the complement of the line segment between ζ and ζ'. Geometrically, this is clear: either the imaginary part of $f(z)$ is non-zero, in which case there is nothing to prove. If the imaginary part of $f(z)$ is zero, then

$$z - \zeta = r(z - \zeta')$$

for some $r \in \mathbb{R}$. Thinking of this in vectors, the vectors $z - \zeta$ and $z - \zeta'$ must have the same direction, because the point *does not* lie on the segment between ζ and ζ'.

10.2. **Exercise (2).** Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise-differentiable closed path. We show that $n(\zeta, \gamma) = 0$ for all $\zeta \in \mathbb{C}$ with $|\zeta| \gg 0$. First, we know that if $\zeta \in \mathbb{C} \setminus \mathrm{Im}(\gamma)$, then

$$\int_\gamma \frac{1}{z - \zeta} \, \mathrm{d}z = n(\zeta, \gamma) \cdot 2\pi i$$

Now, if $|\zeta| \gg 0$, then the quantity

$$\frac{1}{z - \zeta}$$

can be made arbitrarily small, where $z \in \mathrm{Im}(\gamma)$. This means that for $\zeta \gg 0$, we have

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{1}{z - \zeta} \, \mathrm{d}z \right| = |n(\zeta, \gamma)| < \epsilon$$

where $\epsilon > 0$ is any real number. Since $n(\zeta, \gamma)$ is an integer, this implies that

$$n(\zeta, \gamma) = 0$$

for such ζ. This completes the proof.

11. Lecture 14

11.1. **Exercise (2).** Let U be a domain and let γ be a piecewise-differentiable path in U. Let f_n be a sequence of continuous functions on U converging uniformly to f. The claim is that

$$\lim_n \int_\gamma f_n(z) \mathrm{d}z = \int_\gamma f(z) \mathrm{d}z$$

This is a standard analysis statement about uniform convergence.

12. Leture 15

12.1. **Exercise (1).** Uniqueness of $\tilde{f}$ is trivial, because on U', $\tilde{f}$ restricts to f. From there, use the fact that $\tilde{f}$ is continuous.

12.2. **Exercise (2).** Here, we show that the k^{th} order derivative of $(z - c)^k g(z)$ at $z = c$ is $k! g(c)$. By the general Leibniz formula, we have for any $k \geq 0$,

$$\begin{aligned}
[(z-c)^{k+1} g(z)]^{(k+1)}(c) &= \sum_{i=0}^{k+1} \binom{k+1}{i} ((z-c)^{k+1})^{(k+1-i)}(c) g^{(i)}(c) \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} \frac{(k+1)!}{(i)!} (z-c)^i(c) g^{(i)}(c) \\
&= (k+1)! g(c)
\end{aligned}$$

12.3. **Exercise (3).** Suppose $\zeta_2 \in B_{c,R}$ is fixed. By definition,

$$G(\zeta_1, \zeta_2) = \int_\gamma \frac{1}{(z-\zeta_1)(z-\zeta_2)}\,\mathrm{d}z$$

where γ is the path given by $\partial B_{c,R}$. Clearly, function $h : \mathrm{Im}(\gamma) \to \mathbb{C}$ given by

$$h(z) = \frac{1}{(z-\zeta_2)}$$

is continuous. Also, note that

$$G(\zeta_1, \zeta_2) = \int_\gamma \frac{h(z)}{(z-\zeta_1)}\,\mathrm{d}z$$

Lemma 14.2 then implies that $G(\zeta_1, \zeta_2)$ as a function of ζ_1 is a holomorphic function.

12.4. **Exercise (4).** Let $f : U \setminus \{c\} \to \mathbb{C}$ be a non-zero holomorphic function for which c is a removable singularity, where U is an open neighborhood of c. We know that f can be extended to a holomorphic function $\tilde{f}$ on U. Now, $\tilde{f}$ being holomorphic on U is also analytic. Since $\tilde{f}$ is non-zero (because f is non-zero), not all $f^{(k)}(c)$ are zero for $k \in \mathbb{N}$. So, let $k \in \mathbb{N}$ be the smallest integer such that $f^{(k)}(c) \neq 0$. By **Theorem** 15.3 applied to $n = k+1$, we see that

$$f(z) = \frac{f^{(k)}(c)}{k!}(z-c)^k + (z-c)^{k+1} f_{k+1}(z)$$

for $z \in U$, where f_{k+1} is a holomorphic function on U. So,

$$f(z) = (z-c)^k \left(\frac{f^{(k)}(c)}{k!} + (z-c) f_{k+1}(z) \right)$$

and the parenthesis computed at $z = c$ is non-zero. This proves the claim.

12.5. **Exercise (5).** Let U be a domain and let f be a non-zero holomorphic function on U. Since f is analytic on U, its zeroes are isolated. Let C be a compact subset of U. We show that f has only finitely many zeros in C. For the sake of contradiction, suppose f has infinitely many zeroes in C. Let $\{a_n\}$ be a sequence of zeroes in C. Then, this sequence has a convergent subsequence $\{a_{n_k}\}$ in C, and note that since C is compact, the limit also belongs to C. But this contradicts the fact that the zeroes of f are isolated.

Next, suppose $c \in U$ is a zero of f. Again, since the zeroes of f are isolated, there is an open ball around c on which f is non-zero. Since f is analytic, it has a Taylor expansion about the point c. So, it follows that there is some $m > 0$ such that $f^{(m)}(c) \neq 0$ (otherwise the Taylor expansion would imply that f is zero on around the point). Choose the minimal such m. Then just like in the previous problem, we can write

$$f(z) = (z-c)^m f_1(z)$$

for some holomorphic function f_1 on U, such that $f_1(c) \neq 0$. This m is called the *order* of the zero c.

13. Lecture 16

13.1. **Exercise (1).** Let $f(X) = \sum_{i=0}^{d} b_i X^i \in \mathbb{R}[x]$ with $b_d > 0$. Then observe that

$$\lim_{X\to\infty} f(X) = \infty$$

which can be obtained by factoring out the X^d term. This is what we wanted to show.

13.2. **Exercise (2).** Consider the line $x = 1$ in the plane. Any $z \in \mathbb{C}$ on this line is of the form $z = 1 + ib$, where $b \in \mathbb{R}$. In that case, observe that

$$|e^z| = |e^{1+ib}| = |e| \cdot |e^{ib}| = |e|$$

and this is the required counter example for **Lemma** 16.4.

14. Lecture 17

14.1. **Exercise (1).** This is part (3) of **Proposition** 17.4, and it easily follows from part (1) of the same proposition.

14.2. **Exercise (2).** Here, we will give a proof of **Proposition** 17.5. Currently, there is an error in **Proposition** 17.5 as it stands. I think it should be given that $m \leq 0$. A counterexample is the function $f(z) = z$ with $c = 0$. So, I will give a proof assuming $m \leq 0$.

It is clear that (1) $\iff$ (2), because both conditions imply that c is a removable singularity of f.

Now, let us show that (2) $\iff$ (3). Consider the holomorphic extension $\tilde{f}$ of f. Clearly, c is a zero of $\tilde{f}$, and hence has some order $m > 0$. So we can write

$$\tilde{f}(z) = (z - c)^m f_1(z)$$

where f_1 is some holomorphic function on U such that $f_1(c) \neq 0$. So, we see that $m - 1 \geq 0$, and also

$$\frac{\tilde{f}(z)}{(z - c)^{m-1}} = (z - c) f_1(z)$$

which implies that

$$\lim_{z \to c} |z - c|^{1-m} |f(z)| = 0$$

Moreover, if $n = m + 1$, then we see that

$$\lim_{z \to c} |z - c|^{-n} |f(z)| = \infty$$

and this proves the forward direction. To show that (3) $\implies$ (2), observe that if such an $m \leq 0$ exists, then

$$\lim_{z \to c} f(z) = 0$$

and this proves the equivalence.

Finally, let us show that (2) $\iff$ (4). If (2) is true, then put N to be the negative of the order of c as a zero of the extension $\tilde{f}$. It is clear that (4) $\implies$ (2). This completes the proof.

14.3. **Exercise (3).** Here, we will give a proof of **Proposition** 17.6.

The proof of this proposition is very similar to that of the proof given above of **Proposition** 17.5; just consider the function $1/f$, and repeat the steps.

14.4. **Exercise (4).** Let U be a domain, $c \in U$ and f holomorphic on $U \setminus \{c\}$. Suppose c is a pole of f. By **Proposition** 17.4 part (2), there is a positive integer N and a neighborhood V of c in U such that

$$f(z) = \sum_{k=-N}^{\infty} a_k(z-c)^k$$

on $V \setminus \{c\}$. Here $a_{-N} \neq 0$. Let $r > 0$ such that $\overline{B_{c,r}} \subseteq V$. Let $\gamma : [0,1] \to V$ be the path $t \mapsto c + re^{2\pi it}$. By **Exercise** 2 of Lecture 14, we have

$$\int_\gamma f \, \mathrm{d}z = \sum_{k=-N}^{\infty} \int_\gamma a_k(z-c)^k \, \mathrm{d}z = \int_\gamma \frac{a_{-1}}{z-c} \, \mathrm{d}z = a_{-1} 2\pi i$$

and note that all other integrals have vanished because all the other $(z-c)^k$s have primitives on $\mathbb{C} \setminus \{0\}$. This completes the proof.

15. Lecture 20

15.1. **Exercise (1).** The Jacobian of the map $z \to \overline{z}$ is

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Clearly, the above matrix is orthogonal with determinant -1. To, it preserves angles but not orientation.

Exercise (3). Suppose $f : U \to \mathbb{C}$ is conformal, and let $p \in U$. We know that $f'(p) \neq 0$. Now, recall that

$$Jf = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{-\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix}$$

and hence we see that

$$\det(Jf) = \frac{\partial u}{\partial x}^2 + \frac{\partial v}{\partial x}^2$$

Since $f'(p) \neq 0$, we see that $\det(Jf) > 0$. In particular, Jf is invertible. So, one can apply the inverse function theorem to f at the point p. It follows that f maps a neighborhood of p homeomorphically onto its image.

15.2. **Exercise (4).** Let $U \subseteq \mathbb{C}$ be a domain, and let f be an injective holomorphic function on U. We show that f is conformal on U, and it is enough to show that $f'(z) \neq 0$ for any $z \in U$. Consider reading this link for a proof.