ONE DIMENSIONAL RIEMANN INTEGRALS

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(1). Let $S \subset \mathbb{R}$ be bounded above. We show that $a = \sup S$ iff $a \geq x$ for all $x \in S$ and there exists a sequence x_n of elements of S such that $x_n \to a$.

First, suppose $a = \sup S$. By the definition of the supremum, it means that $a \geq x$ for all $x \in S$. Next, let $n \in \mathbb{N}$, and consider the number

$$a - \frac{1}{n}$$

which is strictly less than a. Again, by the definition of the supremum, there exists some $x_n \in S$ such that

$$a - \frac{1}{n} < x_n \leq a$$

and consider the sequence $\{x_n\}$. It is easy to see that $x_n \to a$ because

$$0 \leq a - x_n < a - \left(a - \frac{1}{n}\right) = \frac{1}{n}$$

implying that $|a - x_n| \to 0$ as $n \to \infty$.

Conversely, suppose $a \in \mathbb{R}$ satisfies the given properties. Then, a is an upper bound for S. Let $\{x_n\}$ be a sequence of elements of S converging to a. Let $\epsilon > 0$ be given. So, there is some $n \in \mathbb{N}$ for which

$$0 \leq a - x_n < \epsilon$$

which means that for this n,

$$a - \epsilon < x_n$$

This shows that $a - \epsilon$ cannot be an upper bound for S, for any $\epsilon > 0$. This shows that $a = \sup S$, completing the proof. An analogous statement and proof holds for $\inf S$ as well, if S is assumed to be bounded below.

(2). Let $a, b \in \mathbb{R}$ such that $a < b$. We compute supremums and infimums in the following cases.

(a) $S = [a, b]$. Clearly for all $x \in S$, $x \leq b$. Moreover, $b \in S$, and hence $\sup S = b$, because if a set has a maximum element, then it must be the supremum. A very similar argument shows that $\inf S = a$.

(b) $S = [a, b)$. We have that $x \leq b$ for all $x \in S$ (infact the inequality is strict). Moreover, for any $\epsilon > 0$ such that $a < b - \epsilon$, we see that $b - \epsilon \in S$, so that there is some sequence $\{x_n\}$ of elements in S converging to b. By problem **(1).** we see that $b = \sup S$. The same argument as in **(a)** will show that $\inf S = a$.

(c) $S = (a, b]$. This is symmetric to case **(b)**, we just have an interval open on the left and closed on the right. It follows that $\inf S = a$ and $\sup S = b$.

(d) $S = (a, b)$. For the supremum, the same justification as in **(b)** shows that $\sup S = b$. A similar justification will show that $\inf S = a$.

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(3). In this problem, we compute supremums and infimums of the given sets.

(a) $S = \{x \in \mathbb{Q} | x^2 \leq 2\}$. If $x^2 \leq 2$, and we have that $x \leq \sqrt{2}$. Since x in consideration is rational, it follows that $x < \sqrt{2}$, so that $\sqrt{2}$ is an upper bound (in $\mathbb{R}$). Also, $-\sqrt{2}$ is a lower bound (which is easy to see). Moreover, from Analysis 1, we know that there is a sequence of rationals less than $\sqrt{2}$ converging to $\sqrt{2}$. Eventually, the terms of this sequence of rationals have their squares less than 2, and so applying problem **(1)**, we see that sup $S = \sqrt{2}$. By symmetry of the square function, we see that inf $S = -\sqrt{2}$. Both of these are taken in $\mathbb{R}$. However, since $\mathbb{Q}$ is the set in consideration, it follows that this set has no supremum/infimum.

(b) $S = \{x \in \mathbb{Q} | x^2 < 2\}$. As mentioned in part **(a)**, because only rationals are being considered, this set has no supremum/infimum.

(c) $S = \{x \in \mathbb{Q} | x > 0, x^2 \leq 2\}$. Again, this set has no supremum. However, since only *positive* rationals are being considered, this set is bounded below by 0. Moreover, if $0 < x < 1$, then $x^2 < 1 < 2$, so that there is a sequence of members of S converging to 0. This shows that inf $S = 0$.

(d) $S = \{x \in \mathbb{R} | x > 0, x^2 \leq 2\}$. The infimum, as computed in part (c), is 0. For the supremum, as computed in part **(a)**, we have sup $S = \sqrt{2}$, because $\sqrt{2}$ is infact a real number.

(4). Here, we determine which of the given functions on $(-1, 1)$ are uniformly continuous.

(a) f is defined by

$$f(x) = \begin{cases} 1 & \text{, if } x \geq 0 \\ -1 & \text{, otherwise} \end{cases}$$

Observe that this function is not even continuous at the point $x = 0$, and hence it cannot be uniformly continuous.

(b) $f(x) = x$. Let $\epsilon > 0$ be given, and let $\delta = \epsilon$. If $|x - y| < \delta$, then $|f(x) - f(y)| = |x - y| < \epsilon$, so that f is uniformly continuous.

(c) $f(x) = \tan \frac{\pi x}{2}$. We know that

$$\lim_{x \to 1^-} \tan \frac{\pi x}{2} = \infty$$

and that tan is continuous on $(-1, 1)$. We show that a uniformly continuous function on $(-1, 1)$ cannot be unbounded, which will show that f in our case is not uniformly continuous.

So, consider a function g on $(-1, 1)$ that is uniformly continuous. Let $\epsilon > 0$ be fixed. Then, there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \epsilon$, for $x, y \in (-1, 1)$. Now, take points $x_1 < x_2 < ... < x_n$ in $(-1, 1)$ such that

$$(-1, 1) \subset B(x_1, \delta) \cup ... \cup B(x_n, \delta)$$

which is possible because $(-1, 1)$ is a bounded interval. Then let $M = \max_i |g(x_i)|$. For any $x \in (-1, 1)$, there is an i for which $x \in B(x_i, \delta)$, and in that case

$$|g(x)| < |g(x_i)| + \epsilon \leq M + \epsilon$$

so that g is bounded. This finishes the proof. Hence, f is not uniformly continuous in our case.

(5). Consider the function $\log x$ on $[1, \infty)$. We know that this function is differentiable in the given interval, and

$$\log' x = \frac{1}{x}$$

for all $x \in [1, \infty)$. Observe that

$$0 < \frac{1}{x} \leq 1$$

for all $x \in [1, \infty)$, so that the derivative is bounded. Using this, we show that $\log x$ is uniformly continuous in the given domain. Let $\epsilon > 0$ be given, and let $\delta = \epsilon$. So if $x, y \in [1, \infty)$ such that $|x - y| < \delta$, then observe that

$$|\log x - \log y| = |\log'(c)||x - y| \leq |x - y| < \epsilon$$

where we used the mean value theorem (i.e c is between x and y). Hence, the function is uniformly continuous.

(6). Let f be a continuously differentiable function defined in an open interval. We claim that the following holds: if f' is bounded, then f is uniformly continuous. Let the open interval be (a, b), and suppose $|f'(x)| \leq M$ for all $x \in (a, b)$. Let $\epsilon > 0$ be given, and put $\delta = \epsilon/M$. If $x, y \in (a, b)$ such that $|x - y| < \delta$, then by the mean value theorem we see that

$$|f(x) - f(y)| = |f'(c)||x - y| < M\epsilon/M = \epsilon$$

so that f is uniformly continuous over this open interval. This completes the proof.

Before doing problem **(7)**, I will state the following theorem which I will be using (did not include the proof as the assignment was already too lengthy, and this is a standard theorem).

Theorem: Let f be Riemann integrable on $[a, b]$ such that $m \leq f \leq M$, and let g be a continuous function on $[m, M]$. Then, $g \circ f$ is also Riemann integrable on $[a, b]$.

(7). Here, we prove some basic properties of the Riemann Integral. Throughout, let $I = [a, b]$, and let $f, g : [a, b] \to \mathbb{R}$ be Riemann-Integrable functions.

(a) $f + g$ is also integrable, and

$$\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Let P be a partition of I. The key fact that we will use is

$$L(P, f) + L(P, g) \leq L(P, f + g) \leq U(P, f + g) \leq U(P, f) + U(P, g) \tag{0.1}$$

We only show that inequality for the upper sums, and the inequality for the lower sum has an analogous argument. Suppose P is the partition $a = t_0 < t_1 < ... < t_n = b$. Then,

$$U(P, f + g) = \sum_{i=1}^{n} \sup_{x \in [t_{i-1}, t_i]} (f + g)(x)\Delta t_i \tag{0.2}$$

$$\leq \sum_{i=1}^{n} [\sup_{x \in [t_{i-1}, t_i]} f(x) + \sup_{x \in [t_{i-1}, t_i]} g(x)]\Delta t_i \tag{0.3}$$

$$= U(P, f) + U(P, g) \tag{0.4}$$

where we used the simple fact

$$\sup_{x\in[t_{i-1},t_i]}(f+g)(x) \leq \sup_{x\in[t_{i-1},t_i]} f(x) + \sup_{x\in[t_{i-1},t_i]} g(x)$$

This proves the given inequality. Now, let $\epsilon > 0$ be given. By the Cauchy-criterion for integrability, we find partitions P_1, P_2 of I such that

$$0 \leq U(P_1, f) - L(P_1, f) < \epsilon \tag{0.5}$$

$$0 \leq U(P_2, g) - L(P_2, g) < \epsilon \tag{0.6}$$

and if we put P to be a common refinement of P_1 and P_2, then the above two inequalities hold there as well. So, we get

$$U(P,f) + U(P,g) - L(P,f) - L(P,g) < 2\epsilon \tag{0.7}$$

Observe that

$$L(P, f+g) \leq \underline{\int_a^b}(f+g)(x)dx \leq \overline{\int_a^b}(f+g)(x)dx \leq U(P, f+g) \tag{0.8}$$

Finally, combining $(0.1), (0.7)$ and (0.8), we get

$$\overline{\int_a^b}(f+g)(x)dx - \underline{\int_a^b}(f+g)(x)dx < 2\epsilon$$

since ϵ was arbitrary, this shows that the upper and lower integrals are equal, and hence $f+g$ is integrable. Now to show that the integral is the sum of the two integrals, first observe that

$$L(P,f) + L(P,g) \leq \int_a^b f(x)dx + \int_a^b g(x)dx \leq U(P,f) + U(P,g)$$

and then using $(0.7), (0.8)$ again, we see that

$$\left|\int_a^b f(x)dx + \int_a^b g(x)dx - \int_a^b (f+g)(x)\right| < 2\epsilon$$

and since ϵ is arbitrary, the desired equality follows.

Next, we show that cf is also integrable. If $c = 0$ the claim is trivial. We may also assume without loss of generality that $c > 0$ (if $c < 0$, upper sums will become lower sums and vice-versa, i.e inequalities would be reversed). The key observation here is that for any partition P,

$$U(P, cf) = cU(P, f) \text{ and } L(P, cf) = cL(P, f)$$

which is proved as follows: we have

$$\begin{aligned} U(P, cf) = \sum_{i=1}^n \sup_{x\in[t_{i-1},t_i]}(cf)(x)\Delta t_i &= \sum_{i=1}^n c \sup_{x\in[t_{i-1},t_i]} f(x)\Delta t_i \\ &= cU(P, f) \end{aligned}$$

and this is where the roles are reversed if $c < 0$, i.e supremums will become infimums. So, find a partition P such that

$$U(P,f) - L(P,f) < \frac{\epsilon}{c} \tag{0.9}$$

implying that

$$U(P, cf) - L(P, cf) < \epsilon \tag{0.10}$$

and by Cauchy's criterion for integration, cf is integrable. Finally, observe that

$$L(P, cf) \leq \int_a^b cf dx \leq U(P, cf)$$

implying that

$$cL(P, f) \leq \int_a^b cf dx \leq cU(P, f)$$

and dividing throughout by c, we get

$$L(P, f) \leq \frac{\int_a^b cf dx}{c} \leq U(P, f)$$

and then using (0.9), we have

$$\left| \frac{\int_a^b cf dx}{c} - \int_a^b f dx \right| < \frac{\epsilon}{c}$$

and multipling throughout by c, we see that

$$\left| \int_a^b cf dx - c\int_a^b f dx \right| < \epsilon$$

proving the desired equality, since ϵ was arbitrary.

Finally, we show that fg is also integrable by using the **Theorem** mentioned before the solution to this problem. The map $x \mapsto x^2$ is continuous, and hence f^2 is integrable for any integrable function f (on I). This shows that $(f+g)^2$ is integrable (since $f+g$ is), and so is $(f-g)^2$ and consider the fact that

$$4fg = (f+g)^2 - (f-g)^2$$

showing that fg is integrable (we used both results proved above).

(b) Suppose f, g are integrable on $[a, b]$ such that $f(x) \leq g(x)$ for all $x \in [a, b]$. Let P be any partition of $[a, b]$. Then, we have

$$U(P, f) = \sum_{i=1}^{n} \sup_{x \in [t_{i-1}, t_i]} f(x)\Delta t_i \leq \sum_{i=1}^{n} \sup_{x \in [t_{i-1}, t_i]} g(x)\Delta t_i = U(P, g)$$

and taking the infimum over all partitions, we see that

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

(c) Let f be integrable on $[a, b]$, and let $a < c < b$. We show that f is also integrable on $[a, c]$ and $[c, b]$, and that

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$$

First, let $\epsilon > 0$ be given. Take a partition P such that

$$U(P, f) - L(P, f) < \epsilon$$

Adjoin the point c to P (if it is not already there) to get a refinement P' of P containing c. Still, it holds that

(0.11) $$U(P', f) - L(P', f) < \epsilon$$

Now here is the key observation. Suppose P' is the partition $a = t_0 < ... < t_k = c < t_{k+1} < ... < t_n = b$. Let P_1 be the partition $a = t_0 < ... < t_k = c$ of $[a, c]$ and let P_2 be the partition $c = t_k < t_{k+1} < ... < t_n = b$ of $[c, b]$. Observe that

$$U(P', f) = U(P_1, f|_{[a,c]}) + U(P_2, f|_{[c,b]})$$

and a similar equality holds for lower sums as well. So, we see that

$$U(P', f) - L(P', f) = U(P_1, f|_{[a,c]}) - L(P_1, f|_{[a,c]}) + U(P_2, f|_{[c,b]}) - L(P_2, f|_{[c,b]}) < \epsilon$$

and since upper sum minus the lower sum is always non-negative, this implies

$$U(P_1, f|_{[a,c]}) - L(P_1, f|_{[a,c]})) < \epsilon$$

showing that $f|_{[a,c]}$ is integrable. Same holds for the interval $[c, b]$. To get the desired equality, it is enough to observe that

$$L(P_1, f|_{[a,c]}) + L(P_1, f|_{[c,b]}) \leq \int_a^c f(x)dx + \int_c^b f(x)dx \leq U(P_1, f|_{[a,c]}) + U(P_1, f|_{[c,b]}))$$

and by (0.11), we see that

$$\left| \int_a^b f(x)dx - \int_a^c f(x)dx - \int_c^b f(x)dx \right| < \epsilon$$

proving the claim, since ϵ was arbitrary.

(d) Suppose $f : I \to \mathbb{C}$ is complex valued, with

$$f(x) = f_r(x) + i f_i(x)$$

Suppose f is integrable, i.e both f_r and f_i are integrable on $[a, b]$. By the **Theorem** mentioned before, we see that f_r^2 and f_i^2 are also integrable (since $x \mapsto x^2$ is continuous), and so is $f_r^2 + f_i^2$. Also, the function $x \mapsto \sqrt{x}$ is also continuous in its domain, and hence $\sqrt{f_r^2 + f_i^2}$ is also integrable on $[a, b]$, showing that $|f|$ is also integrable on $[a, b]$.

Next, we show the inequality

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

Let $z = \left(\int_a^b f_r dx, \int_a^b f_i dx \right) = (z_1, z_2)$ be in $\mathbb{R}^2$. Then, observe that

$$\begin{aligned} |z|^2 &= z_1^2 + z_2^2 \\ &= z_1 \int_a^b f_r dx + z_2 \int_a^b f_i dx \\ &= \int_a^b z_1 f_r dx + \int_a^b z_2 f_i dx \\ &= \int_a^b z_1 f_r + z_2 f_i dx \end{aligned}$$

Applying the Cauchy-Schwarz inequality, observe that

$$(z_1 f_r + z_2 f_i)(x) \leq |z||f(x)|$$

and hence, integrating both sides, we get

$$|z|^2 \leq |z| \int_a^b |f(x)|dx$$

Now, if $|z| = 0$, then $z = 0$, and the inequality is trivial. So, assume that $|z| > 0$, and dividing throughout by $|z|$, we get

$$|z| \leq \int_a^b |f(x)|dx$$

which is the desired result.

(8). Throughout, let p, q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

(a) Suppose $u \geq 0$, $v \geq 0$. We show that

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

To show this, fix v, p, q, and define

$$h(u) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

Moreover, we have that

$$h'(u) = u^{p-1} - v$$

and that

$$h''(u) = (p-1)u^{p-2}$$

so that $h''(u) \geq 0$ for all $u \geq 0$ (because $p \geq 1$). This shows that h attains a global minima (where the domain of h is $[0, \infty)$ at the point $u = v^{\frac{1}{p-1}}$. Finally, note that

$$h(v^{\frac{1}{p-1}}) = 0$$

and hence it follows that $h(u) \geq 0$ for all $u \geq 0$. This proves the inequality.

(b) Let f, g be non-negative Riemann-Integrable functions on $[a, b]$. We show that

$$\int_a^b fg dx \leq \left(\int_a^b f^p dx\right)^{\frac{1}{p}} \left(\int_a^b g^q dx\right)^{\frac{1}{q}}$$

We assume that both integrals on the RHS are non-zero (otherwise, we will have to use a measure theoretic argument, which we haven't covered yet).

First consider the case when both the integrals on the RHS are unity. Since f, g are non-negative, we apply the inequality in **(a)**, and get

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

Integrating both sides, we get

$$\int_a^b fg dx \leq \int_a^b \frac{f^p}{p} dx + \int_a^b \frac{g^q}{q} dx = \frac{1}{p} + \frac{1}{q} = 1$$

and hence the inequality is clear in this case.

For the general case, define

$$h(x) = \frac{f(x)}{\left(\int_a^b f^p dx\right)^{\frac{1}{p}}}$$

and similarly

$$h_1(x) = \frac{g(x)}{\left(\int_a^b g^q dx\right)^{\frac{1}{q}}}$$

Then we see that

$$\int_a^b h^p dx = \int_a^b h_1^q dx = 1$$

and hence from the special case, we have

$$\int_a^b hh_1 dx \leq 1$$

However, we have

$$\int_a^b hh_1 dx = \frac{\int_a^b fg dx}{\left(\int_a^b f^p dx\right)^{\frac{1}{p}} \left(\int_a^b g^q dx\right)^{\frac{1}{q}}} \leq 1$$

and from here we get the desired result.

(c) Let f, g be complex integrable functions on $[a, b]$. By problem **(7)** part **(d)**, we know that both $|f|, |g|$ are also Riemann-integrable on $[a, b]$. Applying the result **(c)** to these, we get that

$$\int_a^b |f||g| dx \leq \left(\int_a^b |f|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g|^q dx\right)^{\frac{1}{q}}$$

Again by problem **(7)** part **(d)**, we know that

$$\left|\int_a^b fg dx\right| \leq \int_a^b |f||g| dx$$

and combining these two inequalities, we get the desired results.

(9). Let u be a complex integrable function on $[a, b]$, and we define

$$||u||_2 := \left(\int_a^b u^2 dx\right)^{\frac{1}{2}}$$

We show the triangle-inequality, i.e

$$||f - h||_2 \leq ||f - g||_2 + ||g - h||_2$$

for complex integrable functions f, g, h on $[a, b]$.

We have

$$\begin{aligned}
\int_a^b |f-h|^2 dx &= \int_a^b |f - g + g - h|^2 dx \\
&= \int_a^b |f-g|^2 dx + \int_a^b |g-h|^2 dx + 2\int_a^b |f-g| \cdot |g-h| dx \\
&= ||f-g||_2^2 + ||g-h||_2^2 + 2\int_a^b |f-g| \cdot |g-h| dx \\
&\leq ||f-g||_2^2 + ||g-h||_2^2 + 2||f-g||_2||g-h||_2 \\
&= (||f-g||_2 + ||g-h||_2)^2
\end{aligned}$$

where in the second last step, Holder's inequality was applied with $p = q = 2$. Finally, the desired result is obtained by taking square roots.