ASSIGNMENT-2

SIDDHANT CHAUDHARY
BMC201953

(1). Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Here we make precise the notion that lower Riemann sums increase under refinements and upper Riemann sums decrease under refinements.

Let P be a partition of $[a, b]$, and let P' be a refinement of P. Without loss of generality, we assume that P' has exactly one more point than P, and for more points we can proceed by induction. So, let the partitions be

$$P := a = t_0 < t_1 < ... < t_k < t_{k+1} < ... < t_n = b$$
$$P' := a = t_0 < t_1 < ... < t_k < t'_k < t_{k+1} < ... < t_n = b$$

i.e, the new point in P' is t'_k, for some $0 \leq k \leq n-1$. For a subinterval $[t_{i-1}, t_i]$, put

$$m_i(f) = \inf_{x \in [t_{i-1}, t_i]} f(x)$$
$$M_i(f) = \sup_{x \in [t_{i-1}, t_i]} f(x)$$

and put

$$M_1 = \sup_{x \in [t_k, t'_k]} f(x)$$
$$M_2 = \sup_{x \in [t'_k, t_{k+1}]} f(x)$$
$$m_1 = \inf_{x \in [t_k, t'_k]} f(x)$$
$$m_2 = \inf_{x \in [t'_k, t_{k+1}]} f(x)$$

So, we have

$$\begin{aligned} U(P, f) - U(P', f) &= M_{k+1}(f)(t_{k+1} - t_k) - M_1(t'_k - t_k) - M_2(t_{k+1} - t'_k) \\ &\geq M_{k+1}(f)(t_{k+1} - t_k) - M_{k+1}(f)(t'_k - t_k) - M_{k+1}(f)(t_{k+1} - t'_k) \\ &= 0 \end{aligned}$$

where we have used the simple fact that $M_1, M_2 \leq M_{k+1}(f)$. Similarly, we have

$$\begin{aligned} L(P, f) - L(P', f) &= m_{k+1}(f)(t_{k+1} - t_k) - m_1(t'_k - t_k) - m_2(t_{k+1} - t'_k) \\ &\leq m_{k+1}(f)(t_{k+1} - t_k) - m_{k+1}(f)(t'_k - t_k) - m_{k+1}(f)(t_{k+1} - t'_k) \\ &= 0 \end{aligned}$$

where we used the simple fact that $m_1, m_2 \geq m_{k+1}(f)$. This shows that upper sums decrease, and lower sums increase by taking refinements.

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(2). Let $I = [a, b]$, and let f be a monotonic function on I. Without loss of generality, we assume that f is monotonic increasing on I (the decreasing case is similar). We show that f is Riemann integrable on I.

Let $\epsilon > 0$ be given, and there exists a $k > 0$ such that

$$k(f(b) - f(a)) < \epsilon$$

Let P be a partition of $[a, b]$ given by

$$P := a = t_0 < t_1 < ... < t_n = b$$

such that $t_i - t_{i-1} < k$ for each $1 \leq i \leq n$. We use the notation as in problem **(1).** Since f is monotonic increasing, we have

$$m_i(f) = f(t_{i-1})$$
$$M_i(f) = f(t_i)$$

So, for this partition we have

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{i=1}^{n} [f(t_i) - f(t_{i-1})](t_i - t_{i-1}) \\ &\leq k \sum_{i=1}^{n} (f(t_i) - f(t_{i-1})) \\ &= k(f(b) - f(a)) \\ &< \epsilon \end{aligned}$$

and hence by the Cauchy-criterion for integrability, this shows that f is integrable on $[a, b]$.

(4). Let X be a metric space, and let $E \subset X$. We show that

$$\partial E = \partial E^c$$

First, suppose $x \in \partial E$. This means that for every $\delta > 0$, $B(x, \delta)$ contains a point of E and a point of E^c. Now, to cases are possible:

(1) $x \in E$. In this case, $x \notin E^c$, and hence x is a limit point of E^c (because every neighborhood contains a point of E^c). So, $x \in \overline{E^c}$. However, $x \notin$ Int(E^c) (as $x \notin E^c$), and hence $x \in \partial E^c$.

(2) $x \notin E$. In this case, $x \in E^c$, so that $x \in \overline{E^c}$. However, every neighborhood of x contains a point of E, and hence $x \notin$ Int(E^c), and hence $x \in \partial E^c$.

So in any case, we see that $\partial E \subset \partial E^c$. We can reverse the roles of E and E^c to get the reverse inclusion, and hence it follows that $\partial E = \partial E^c$.

Before doing the next problem, we prove a lemma.

Lemma 0.1. *Any compact subset of* $\mathbb{R}^2$ *which has content* 0 *is acceptable and has area* 0.

Proof: Let $T \subset \mathbb{R}^2$ be compact with content 0. Since $\partial T \subset T$ (because T is closed), this means that ∂T has content 0, and hence T is acceptable. Now let $\epsilon > 0$ be given , and let $R_1, ..., R_n$ be a collection of closed rectangles in $\mathbb{R}^2$ such that $T \subset R_1 \cup R_2 \cup ... \cup R_n$ and

$$\sum_{i=1}^{n} \text{area}(R_i) < \epsilon$$

Take a rectangle $R \subset \mathbb{R}^2$ which contains $R_1 \cup ... \cup R_n$. So,

$$\begin{aligned}
\text{area}(T) &= \int_R \chi_T \\
&\leq \int_R \chi_{R_1} + ... + \chi_{R_n} \\
&= \int_R \chi_{R_1} + ... + \int_R \chi_{R_n} \\
&= \sum_{i=1}^{n} \text{area}(R_i) \\
&< \epsilon
\end{aligned}$$

and since ϵ was arbitrary, this shows that area$(T) = 0$, completing the proof.

(5). This is just **Lemma** 0.1.

(6). Let R be the rectangle

$$R := \{(x, y) : |x| \leq 2, y \leq 2\}$$

and let

$$S := \{(x, y) : |x| \leq 2, |y| \leq 1\}$$

(a) ∂S : First, observe that S is a closed rectangle in $\mathbb{R}^2$ with vertices

$$(2, 1), (2, -1), (-2, 1), (-2, -1)$$

Hence, the boundary of S in $\mathbb{R}^2$ will just be the four sides of this rectangle, because every other point is an interior point, because we can take a small enough ball which is contained entirely inside the rectangle. So,

$$\partial S = A \cup B \cup C \cup D$$

where

$$\begin{aligned}
A &= \{t(2, 1) + (1 - t)(2, -1) : t \in [0, 1]\} \\
B &= \{t(2, -1) + (1 - t)(-2, -1) : t \in [0, 1]\} \\
C &= \{t(-2, -1) + (1 - t)(-2, 1) : t \in [0, 1]\} \\
D &= \{t(-2, 1) + (1 - t)(2, 1) : t \in [0, 1]\}
\end{aligned}$$

(b) ∂R : This is similar to part **(a)**, because R is a closed rectangle in $\mathbb{R}^2$ with vertices

$$(2, 2), (2, -2), (-2, 2), (-2, -2)$$

and hence the boundary of R in $\mathbb{R}^2$ will be the four sides of R. Hence,

$$\partial R = A' \cup B' \cup C' \cup D'$$

where

$$\begin{aligned}
A' &= \{t(2, 2) + (1 - t)(2, -2) : t \in [0, 1]\} \\
B' &= \{t(2, -2) + (1 - t)(-2, -2) : t \in [0, 1]\} \\
C' &= \{t(-2, -2) + (1 - t)(-2, 2) : t \in [0, 1]\} \\
D' &= \{t(-2, 2) + (1 - t)(2, 2) : t \in [0, 1]\}
\end{aligned}$$

(c) $\partial(S,R)$: Now, our metric space in question is R. So, $\partial(S,R) \subset \partial(S)$. Here, we claim that $\partial(S,R)$ is the union of the upper and lower edges of the rectangle S, i.e

$$\partial(S,R) = \{t(-2,1) + (1-t)(2,1) : t \in [0,1]\} \cup \{t(2,-1) + (1-t)(-2,-1) : t \in [0,1]\}$$

This is because any other point of S is an interior point of S wrt R being the metric space, as points outside R are not considered.

Before doing problem **(7).**, we will prove a lemma.

Lemma 0.2. *Let l be a line segment in $\mathbb{R}^2$. Then, l has content 0.*

Proof: If $a,b \in \mathbb{R}^2$, we denote the line segment with endpoints a,b by ab. Let $\epsilon > 0$ be given, and let $p_i, p_f \in \mathbb{R}^2$ be the end points of l. Pick $n \in \mathbb{N}$ be such that

$$\frac{|p_i p_f|}{n} < \epsilon$$

Divide the line segment l into n line-segments, i.e we pick points $p_1, p_2, ..., p_{n-1}$ on l such that

$$l = p_i p_f = p_i p_1 \cup p_1 p_2 \cup ... \cup p_{n-1} p_j$$

and that

$$|p_i p_1| = \frac{|p_i p_f|}{n}$$
$$|p_t p_{t+1}| = \frac{|p_i p_f|}{n} \text{ for each } 1 \le t \le n-2$$
$$|p_{n-1} p_f| = \frac{|p_i p_f|}{n}$$

Put $p_0 = p_i$ and $p_n = p_f$. For any two consecutive points p_j, p_{j+1} on l ($0 \le j \le n-1$), pick a rectangle R_j such that $p_j p_{j+1}$ is one of the diagonals of R_j. Observe that

$$\text{area}(R_j) \le |p_j p_{j+1}|^2 = \frac{|p_i p_f|^2}{n^2}$$

Also, any two rectangles R_j, R_{j+1} intersect only along part of an edge. So, we see that

$$\text{area}(R_0 \cup R_1 \cup ... \cup R_{n-1}) = \text{area}(R_0) + ... + \text{area}(R_{n-1}) = n\frac{|p_i p_f|^2}{n^2} = \frac{|p_i p_f|^2}{n} < \epsilon$$

and that

$$l \subset R_0 \cup ... \cup R_{n-1}$$

So, we have covered l with finitely many rectangles with total area less than ϵ, implying that l has content zero. This completes the proof.

(7). Let $S = L \cup R \subset \mathbb{R}^2$ where

$$L = \{(x,0) : 1 \le |x| \le 2\}, R = \{(x,y) : |x| \le 1, |y| \le 1\}$$

We can write

$$L = \{(x,0) : 1 \le x \le 2\} \cup \{(x,0) : -2 \le x \le -1\} = l_1 \cup l_2$$

and hence L is a disjoint union of two segments.

First, we show that S is acceptable. We claim that

$$\partial S = L \cup \text{ the four sides of the rectangle } R$$

which is simply because every neighborhood of each point in this set intersects with S^c. Now,

$$L \cup \text{ the four sides of the rectangle } R$$

is a finite union of line segments in $\mathbb{R}^2$, and by the previous **Lemma** 0.2, we know that each line segment has content 0, and that this finite union has content 0. Hence, ∂S has content 0. Finally, since S is closed and bounded, it is compact, and hence S is *acceptable*.

Next, we compute the area of S. Consider the rectangle $R' = [-2, 2] \times [-2, 2]$, which contains S. We have

$$\begin{aligned}
\text{area}(S) = \int_{R'} \chi_S &= \int_{R'} \chi_R + \chi_L - \chi_{\{(1,0),(-1,0)\}} \\
&= \int_{R'} \chi_R + \int_{R'} \chi_L - \int_{R'} \chi_{\{(1,0),(-1,0)\}} \\
&= \text{area}(R) + \int_{R'} \chi_{l_1} + \int_{R'} \chi_{l_2} - \int_{R'} \chi_{\{(1,0),(-1,0)\}} \\
&= \text{area}(R) + \text{area}(l_1) + \text{area}(l_2) - \text{area}(\{(1,0),(0,1)\}) \\
&= 4
\end{aligned}$$

We have used the fact that any line segment is compact and by **Lemma** 0.2, it has content 0, and so by **Lemma** 0.1 it has area 0. The same thing holds for a finite set as well.

(8). Here, we will derive the formula for the area of a triangle. We will assume that area is preserved under rotations and translations.

First, we find the formula for a right angled triangle, since every triangle can be broken into two right triangles intersecting along an edge. So without loss of generality, let the vertices of the triangle T be

$$(0,0), (a,0), (0,b)$$

Let D be the segment with endpoints $(0, b)$ and $(a, 0)$. Now, the boundary of this triangle is the union of the three segments

$$\begin{gathered}
\{t(0,0) + (1-t)(a,0) : t \in [0,1]\} \\
\{t(0,0) + (1-t)(0,b) : t \in [0,1]\} \\
\{t(a,0) + (1-t)(0,b) : t \in [0,1]\}
\end{gathered}$$

and by **Lemma** 0.2, each of these segments has content 0, and hence their union has content 0. This shows that the boundary of the triangle has content 0, and hence the triangle being compact is acceptable.

Now, let R be the rectangle $[0, a] \times [0, b]$, and let P be the partition of R where the subrectangles are of the form

$$\left[\frac{ak_1}{n}, \frac{a(k_1+1)}{n}\right] \times \left[\frac{bk_2}{n}, \frac{b(k_2+1)}{n}\right]$$

where $0 \le k_1, k_2 \le n-1$. In other words, we are uniformly dividing the intervals $[0, a]$ and $[0, b]$ into n sub-intervals of sizes $\frac{a}{n}$ and $\frac{b}{n}$ respectively. Now we will see how upper and lower sums $U(P, \chi_T)$ and $L(P, \chi_T)$ behave.

First, observe that if $k_1 + k_2 \le n$, then the point

$$\left(\frac{ak_1}{n}, \frac{bk_2}{n}\right)$$

lies on or below the segment D. This means that if S is any subrectangle of the form

$$S = \left[\frac{ak_1}{n}, \frac{a(k_1+1)}{n}\right] \times \left[\frac{bk_2}{n}, \frac{b(k_2+1)}{n}\right]$$

with $k_1+k_2 \leq n$, then the supremum of χ_T over S is 1. Otherwise, the supremum is 0. So, we see that

$$U(P,f) = \sum_{k_1+k_2\leq n} \text{area}\left(\left[\frac{ak_1}{n}, \frac{a(k_1+1)}{n}\right] \times \left[\frac{bk_2}{n}, \frac{b(k_2+1)}{n}\right]\right) = \sum_{k_1+k_2\leq n} \frac{ab}{n^2}$$

Now observe that in the above sum, $0 \leq k_1 \leq n-1$, and for every such k_1, there are $n-k_1$ possible values of k_2 such that $k_1+k_2 \leq n$. So,

$$\begin{aligned}
\sum_{k_1+k_2\leq n} \frac{ab}{n^2} &= \frac{ab}{n^2} \sum_{k_1+k_2\leq n} 1 \\
&= \frac{ab}{n^2} \sum_{k_1=0}^{n-1} (n-k_1) \\
&= \frac{ab}{n^2}\left(n_2 - \frac{(n-1)n}{2}\right) \\
&= \frac{ab}{n^2}\frac{n^2-n}{2}
\end{aligned}$$

Now, observe that we have already shown that χ_T is integrable on R (since T is acceptable), and hence we can take the limit as $n \to \infty$ in the upper sums, and they will converge to the integral. So,

$$\int_R \chi_T = \lim_{n\to\infty} \frac{ab(n^2-n)}{2n^2} = \frac{ab}{2}$$

which is the required formula.

Now, suppose T is any general triangle, and by suitable rotations and translations, suppose two points of T lie on the x-axis, and one point on the y-axis (as we can split a triangle into two right angled ones). Let the vertices be

$$(c,0),(a,0),(0,b)$$

where say $a \geq 0$ and $c < 0$. Let l be the line whose end points are $(0,b)$ and $(0,0)$, i.e l is a perpendicular in T. Let T_1 be the triangle with endpoints

$$(0,b),(0,0),(a,0)$$

and T_2 be the triangle with endpoints

$$(0,b),(0,0),(c,0)$$

so that $T = T_1 \cup T_2$ and $T_1 \cap T_2 = l$. As before, T is acceptable because the boundary is a union of three line segments, and has content zero. Let R be any rectangle containing T. So,

$$\begin{aligned}\text{area}(T) &= \int_R \chi_T \\ &= \int_R \chi_{T_1} + \chi_{T_2} - \chi_l \\ &= \int_R \chi_{T_1} + \int_R \chi_{T_2} - \int_R \chi_l \\ &= \text{area}(T_1) + \text{area}(T_2) - \text{area}(l) \\ &= \frac{ab}{2} + \frac{(-c)b}{2} + 0 \\ &= \frac{(a-c)b}{2}\end{aligned}$$

where we used the fact that the area of a line segment is 0 (which follows by **Lemma** 0.1 and **Lemma** 0.2) and the formula for the area of a right triangle. Note that the factor $-c$ was used, as $c < 0$ was assumed, and we computed the formula of the area when all the coordinates are positive. This is the usual base-height formula for the area of a triangle.

(9). Here, we will show that

$$\text{area}(D^2) = \pi$$

where D^2 is the closed disk (of radius 1) in $\mathbb{R}^2$.

First, we show that D^2 is acceptable. Observe that

$$\partial(D^2) = S^1$$

as any neighborhood of a point on the unit circle S^1 intersects $(D^2)^c$, and hence is a boundary point. Every other point is an interior point.

Next, we show that S^1 has content 0. First, we take n equidistant points on the unit circle, say points of the form

$$\left(\cos\frac{2\pi k}{n}, \sin\frac{2\pi k}{n}\right)$$

for $0 \le k \le n-1$. Now, we cover S^1 with rectangles as follows. Consider the diagram given on the next page.

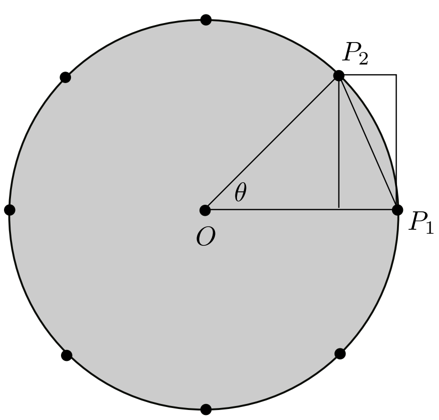

Figure 1. The covering rectangle

Here, P_1 and P_2 are two consecutive points, and $\theta = \dfrac{2\pi}{n}$. Consider the rectangle, one of whose diagonals is P_1P_2. Clearly, all such rectangles will cover S^1, and there will be n such rectangles. The length of P_1P_2 is given by

$$|P_1P_2| = 2\sin\frac{\theta}{2} = 2\sin\frac{\pi}{n}$$

and hence the area of this rectangle is bounded by

$$\text{area} \leq |P_1P_2|^2 = 4\sin^2\frac{\pi}{n}$$

Since there are n rectangles, the total area of these is bounded by

$$4n\sin^2\frac{\pi}{n}$$

Now, observe that

$$\begin{aligned}
\lim_{n\to\infty} 4n\sin^2\frac{\pi}{n} &= 4\lim_{n\to\infty}\frac{\sin^2\pi/n}{1/n} \\
&= 4\pi^2\lim_{n\to\infty}\frac{\sin^2\pi/n}{\pi^2/n^2}\cdot\frac{1}{n} \\
&= 4\pi^2\lim_{n\to\infty}\frac{\sin^2\pi/n}{(\pi/n)^2}\cdot\lim_{n\to\infty}\frac{1}{n} \\
&= 0
\end{aligned}$$

This means no matter what $\epsilon > 0$ is, there is some $n > 0$ such that

$$4n\sin^2\frac{\pi}{n} < \epsilon$$

and hence finitely many rectangles over total area less than ϵ cover S^1, showing that S^1 has content zero. This shows that D^2 is *acceptable*, because it is compact, being closed and bounded.

Now, we calculate area(D^2). First, consider a regular polynomial with n-sides inscribed in this circle (as we did above, taking n equidistant points). Since the inscribed regular polygon is a subset of D^2, we have

$$\text{area}(\text{polygon}) \leq \text{area}(D^2)$$

Now, the polygon can be written as a union of n triangles, and since the triangles only share edges, we can sum the areas of the triangles to obtain the total area (just like we did in **(8).**). The base of one triangle is

$$2\sin\frac{\theta}{2} = 2\sin\frac{\pi}{n}$$

and the height is

$$\cos\frac{\theta}{2} = \cos\frac{\pi}{n}$$

Hence, the area of one triangle is

$$\sin\frac{\pi}{n}\cos\frac{\pi}{n}$$

and since there are n triangles, we have

$$\text{area}(\text{polygon}) = n\sin\frac{\pi}{n}\cos\frac{\pi}{n} \leq \text{area}(D^2)$$

Taking limits as $n \to \infty$, we see that

$$\begin{aligned}\lim_{n\to\infty} n \sin\frac{\pi}{n}\cos\frac{\pi}{n} &= \lim_{n\to\infty}\frac{\sin \pi/n}{1/n}\cdot\cos\frac{\pi}{n}\\ &= \pi \lim_{n\to\infty}\frac{\sin\pi/n}{\pi/n}\cdot\lim_{n\to\infty}\cos\frac{\pi}{n}\\ &= \pi\\ &\le \text{area}(D^2)\end{aligned}$$

Next, we will use regular polygons which *circumscribe* D^2. Suppose there is such a regular polygon with n-sides, and let P_1, P_2 be consecutive points. Then $\angle P_1OP_2 = \theta = \dfrac{2\pi}{n}$, and P_1P_2 is tangent to the circle. So, the height of the triangle in this case is 1 (radius of the disk), and the base is of length

$$2\tan\frac{\theta}{2}$$

So, the area of one triangle is

$$\tan\frac{\theta}{2} = \tan\frac{\pi}{n}$$

and hence the total area of the polygon is

$$n\tan\frac{\pi}{n}$$

Since D^2 is a subset of this polygon, we see that

$$\text{area}(D^2) \le \text{area}(\text{polygon}) = n\tan\frac{\pi}{n}$$

Taking limits as $n \to \infty$, we see that

$$\begin{aligned}\text{area}(D^2) &\le \lim_{n\to\infty} n\tan\frac{\pi}{n}\\ &= \lim_{n\to\infty} n\frac{\sin\pi/n}{\cos\pi/n}\\ &= \pi\lim_{n\to\infty}\frac{\sin\pi/n}{\pi/n}\cdot\lim_{n\to\infty}\frac{1}{\cos\pi/n}\\ &= \pi\end{aligned}$$

and hence we conclude that

$$\text{area}(D^2) = \pi$$

In this problem, we have extensively used limits of trigonometric functions.

(10). Here, we consider *Jordan Measurable* sets, instead of the more stronger *acceptable sets*. While is it true that most of the theory of integration is valid for Jordan Measurable sets too, one of the fundamental facts regarding the integrability of continuous functions is no longer true, i.e given a continuous function on a Jordan Measurable set, it is not necessary that the function is integrable on a rectangle containing the set as well.

For example, observe the following: consider the function interval $(0, 1)$ in $\mathbb{R}$, which is bounded (but not closed) and Jordan Measurable, as it has empty boundary, and hence the measure of the boundary is zero. Consider the function

$$f(x) = \frac{1}{x}$$

on this interval, which is continuous, and also unbounded. Consider the rectangle $[0, 1]$ in $\mathbb{R}$, which contains $(0, 1)$. Clearly, we cannot apply the Tietze extension theorem here. In fact, there is no continuous extension for this function, as it is unbounded. Hence, we cannot integrate this function on $[0, 1]$ in the Riemann sense, and this is one of the properties that fails to go through for Jordan Measurable sets.