ASSIGNMENT-3

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(1). (Problem 3-23 of Spivak). Let A, B be rectangles and let $C \subset R = A \times B$ be a set of content zero. For $x \in A$, let $B_x \subset B$ be defined by

$$B_x = \{y \in B \mid (x, y) \in C\}$$

Let $A' \subset A$ be the set

$$\{x \in A \mid B_x \text{ is not of content zero}\}$$

Show that A' is a set of measure zero.

Solution. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be rectangles, and let $C \subset A \times B$ be a set of content 0. For any $x \in A$, define

$$B_x = \{y \in B : (x, y) \in C\}$$

and define

$$A' = \{x \in A : B_x \text{ is not of content 0}\}$$

We will show that A' has measure 0.

First, we show that we can assume without loss of generality that C is closed. To show this, consider $\overline{C}$. Let $R_1, ..., R_k$ be closed rectangles in $\mathbb{R}^{n+m}$ such that

$$C \subset R_1 \cup ... \cup R_k$$

and

$$\sum_{i=1}^{k} \text{volume}(R_i) < \epsilon$$

Now, $R_1 \cup ... \cup R_k$ is closed, and this means that

$$\overline{C} \subset R_1 \cup ... \cup R_k$$

and hence $\overline{C}$ also has content zero. Moreover, define

$$B'_x = \{y \in B : (x, y) \in \overline{C}\}$$

and also

$$A'' = \{x \in A | B'_x \text{ does not have content 0}\}$$

Then, it is easy to see that $B_x \subset B'_x$ and $A' \subset A''$. So, it is fair to assume that C is closed, and we will do so for the rest of the solution. Moreover, since C is bounded, it is compact.

Since C has content zero, ∂C also has content zero, so that χ_C is integrable on $A \times B$. We showed in **ASSIGNMENT-2** that any compact subset of $\mathbb{R}^k$ of content zero has volume zero, and hence

$$\int_{A \times B} \chi_C = 0$$

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Now, applying Fubini's theorem, we see that

$$\int_{A\times B}\chi_C = \int_A \mathcal{L} = \int_A \mathcal{U} = 0$$

where $\mathcal{L}$,$\mathcal{U}$ are defined on A as

$$\mathcal{U}(x) = \overline{\int_B}\chi_C(x,y)dy = \overline{\int_B}\chi_{C,x}(y)dy$$
$$\mathcal{L}(x) = \underline{\int_B}\chi_C(x,y)dy = \underline{\int_B}\chi_{C,x}(y)dy$$

Also, $\mathcal{U}$ is a non-negative function. We claim that if $x\in A'$, then

$$\mathcal{U}(x) > 0$$

First, let $\pi_y : A\times B \to B$ be the projection map, which is continuous. Observe that for any $x\in A$, we have

$$B_x = \pi_y(C\cap\{x\}\times B)$$

and hence B_x is compact, because $C\cap\{x\}\times B$ is compact. Now, let $x\in A'$, and suppose $\mathcal{U}(x)=0$, which implies that

$$\int_B \chi_{C,x}(y) = 0$$

Now, $\chi_{C,x}$ is a non-negative function on B. Since it is integrable, any point $y\in B$ where $\chi_{C,x}(y)>0$ must be a point of discontinuity. So, it follows that the set of points where $\chi_{C,x}$ is positive has measure 0. But, this set is precisely B_x. Since B_x is compact, measure 0 implies content 0. But, this contradicts the fact that $x\in A'$. So, it must be true that $\mathcal{U}(x)>0$.

Finally, since $\mathcal{U}$ is integrable on A and is non-negative, any point where $\mathcal{U}$ is positive must be a point of discontinuity. By what we have showed above, all points of A' are points of discontinuity of $\mathcal{U}$. Since $\mathcal{U}$ is integrable, this implies that A' has measure 0, completing the proof.

(2). Let $I_i\subset\mathbb{R}$ for $1\leq i\leq n$ be closed bounded intervals of non-zero length. Prove that I_i is *not* of content zero, and an induction to show that $I_1\times ...\times I_n$ is not of measure zero.

Solution. In **ASSIGNMENT-2**, I showed that a compact set in $\mathbb{R}^n$ with content 0 must have volume zero. So, I will show by induction that

$$\text{volume}(I_1\times ...\times I_n) > 0$$

which will show that $I_1\times ...\times I_n$ cannot have content zero. This will be the proof strategy.

For the base case, let $n=1$ and let $I_1=[a_1,b_1]$. Then, we have

$$\text{volume}(I_1) = \int_{a_1}^{b_1} 1 = b_1 - a_1 > 0$$

and clearly the base case is true. For the inductive case, let $I_1=[a_1,b_1],...,I_n=[a_n,b_n]$ be closed and bounded intervals in $\mathbb{R}$ with non-zero length such that

$$\text{volume}(I_1\times ...\times I_n) = (b_1-a_1)...(b_n-a_n) > 0$$

Let $I_{n+1} = [a_{n+1}, b_{n+1}]$ be another closed bounded interval of non-zero length. So, we have

$$\text{volume}(I_1 \times .. \times I_{n+1}) = \int_{[a_1,b_1]\times..\times[a_{n+1},b_{n+1}]} 1 = \int_{[a_{n+1},b_{n+1}]} \int_{[a_1,b_1]\times...\times[a_n,b_n]} 1$$

where we have used Fubini's Theorem above. By inductive hypothesis,

$$\int_{[a_1,b_1]\times...\times[a_n,b_n]} 1 = (b_1 - a_1)...(b_n - a_n)$$

and hence

$$\begin{aligned}\int_{[a_{n+1},b_{n+1}]} \int_{[a_1,b_1]\times...\times[a_n,b_n]} 1 &= \int_{[a_{n+1},b_{n+1}]} (b_1 - a_1)...(b_n - a_n) \\ &= (b_1 - a_1)...(b_n - a_n) \int_{[a_{n+1},b_{n+1}]} 1 \\ &= (b_1 - a_1)...(b_n - a_n)(b_{n+1} - a_{n+1}) \\ &> 0\end{aligned}$$

and by induction, the statement is true for all $n \in \mathbb{N}$. So, every rectangle in $\mathbb{R}^n$ has non-zero volume, and hence it is *not* of content zero.

(3). Let $I = [a, b]$ and f a continuous real-valued function on the square $I \times I$. Prove that

$$\int_a^b \left(\int_a^y f(x, y)dx \right) dy = \int_a^b \left(\int_x^b f(x, y)dy \right) dx$$

Solution. Consider the rectangle $I^2 = [a, b] \times [a, b]$ in $\mathbb{R}^2$, and let $T \subset I^2$ be the triangle

$$T := \{(x, y) \in I^2 | x \leq y\}$$

Clearly, ∂T has measure 0 being a union of three line segments in $\mathbb{R}^2$, and hence T is Jordan Measurable (infact, it is an acceptable set, because it is compact), so that χ_T is integrable on I^2. Since f is a continuous function on I^2, it is integrable over T. Moreover, we have

$$\int_T f = \int_{I^2} f \cdot \chi_T$$

Now, we use Fubini's theorem on the integral in the RHS of the above equation. By Fubini's Theorem, we know that

$$\int_{I^2} f \cdot \chi_T = \int_a^b \mathcal{U}(x)dx$$

where

$$\mathcal{U}(x) = \overline{\int_a^b} f(x, y)\chi_T(x, y)dy = \int_x^b f(x, y)dy$$

and hence we get

$$(\dagger) \qquad \int_T f = \int_a^b \left(\int_x^b f(x, y)dy \right) dx$$

Similarly, by restricting the function to the x-axis instead, we get

$$\int_{I^2} f \cdot \chi_T = \int_a^b \mathcal{U}'(y)dy$$

where

$$\mathcal{U}'(y) = \overline{\int_a^b} f(x,y)\chi_T(x,y)dx = \int_a^y f(x,y)dx$$

and hence

(∗) $$\int_T f = \int_a^b \left(\int_a^y f(x,y)dx \right) dy$$

and by (†) and (∗), we get

$$\int_a^b \left(\int_a^y f(x,y)dx \right) dy = \int_a^b \left(\int_x^b f(x,y)dy \right) dx$$

(4). (Equality of mixed partial derivatives using Fubini!) Let f be a $\mathscr{C}^2$ function on an open rectangle in $\mathbb{R}^2$, with x, y being the coordinates. This means that the partial derivatives upto order two exist and are continuous. Use Fubini to prove that the mixed partials are equal.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Solution. Let U be an open subset of $\mathbb{R}^2$, and suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a $\mathscr{C}^2$ function on U. We show that for any point $(x,y) \in U$, we have

$$D_{21}f(x,y) = D_{12}f(x,y)$$

i.e the second order mixed partials are equal. For the sake of contradiction, suppose there is some $(x,y) \in U$ such that

$$D_{21}f(x,y) - D_{12}f(x,y) > 0$$

Since $D_{21}f - D_{12}f$ is assumed to be continuous, there is a small rectangle $[a,b] \times [c,d]$ containing (x,y) and some $\epsilon > 0$ such that

$$\inf_{(s,t)\in[a,b]\times[c,d]} D_{21}f(s,t) - D_{12}f(s,t) \geq \epsilon$$

and consequently

(∗∗) $$\int_{[a,b]\times[c,d]} D_{21}f - D_{12}f > 0$$

Now, we compute

$$\int_{[a,b]\times[c,d]} D_{21}f - D_{12}f = \int_{[a,b]\times[c,d]} D_{21}f - \int_{[a,b]\times[c,d]} D_{12}f$$

using Fubini's theorem to arrive at a contradiction. First, restricting the function $D_{21}f$ to the y-axis, we have by Fubini's Theorem

$$\int_{[a,b]\times[c,d]} D_{21}f = \int_a^b \mathcal{U}(x)$$

where

$$\mathcal{U}(x) = \overline{\int_c^d} D_{21}f(x,y)dy = \int_c^d D_{21}f(x,y)dy$$

because $D_{21}f$ is continuous. Computing further, we see that

$$\int_c^d D_{21}f(x,y)dy = D_1f(x,d) - D_1f(x,c)$$

by the Fundamental Theorem of Calculus in one variable. So, we obtain

$$\begin{aligned}\int_{[a,b]\times[c,d]} D_{21}f &= \int_a^b (D_1f(x,d) - D_1f(x,c))dx \\ &= \int_a^b D_1f(x,d)dx - \int_a^b D_1f(x,c)dx \\ (\dagger) \qquad &= f(b,d) - f(a,d) - (f(b,c) - f(a,c))\end{aligned}$$

By a very similar strategy, i.e by restricting the function $D_{12}f$ to the x-axis and using Fubini's Theorem, we can obtain

$$(*) \qquad \int_{[a,b]\times[c,d]} D_{12}f = f(b,d) - f(a,d) - (f(b,c) - f(a,c))$$

and hence by $(\dagger)$ and $(*)$ we get that

$$\int_{[a,b]\times[c,d]} D_{21}f - D_{12}f = 0$$

which contradicts equation $(**)$. So, it must be true that

$$D_{21}f(x,y) - D_{12}f(x,y) = 0$$

for all $(x,y) \in U$, and this completes the proof.

(5). Let $R = [a_1,b_1]\times[a_2,b_2] \subset \mathbb{R}^2$ and f a continuous real-valued function defined on R. Define the function F on R by

$$F(x,y) = \int_{[a_1,x]\times[a_2,y]} f$$

Is F continuous? Is it $\mathscr{C}^1$? What are the partial derivatives $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$?

Solution. First, let us show that F is a continuous function on the rectangle R. First, put

$$M = \sup_R |f|$$

By Fubini's Theorem, we see that

$$F(x,y) = \int_{a_1}^x \int_{a_2}^y f$$

Let $a_1 \le x \le b_1$ and $a_2 \le y \le b_2$ be fixed. Let (h,k) be such that $h,k \ge 0$. We have the following equations, for small enough h,k:

$$\begin{aligned}|F(x+h,y+k) - F(x,y)| &= \left|\int_{a_1}^{x+h}\int_{a_2}^{y+k} f - \int_{a_1}^{x}\int_{a_2}^{y} f\right| \\ &= \left|\int_{a_1}^{x+h}\int_{a_2}^{y} f + \int_{a_1}^{x+h}\int_{y}^{y+k} f - \int_{a_1}^{x}\int_{a_2}^{y} f\right| \\ &= \left|\int_{a_1}^{x}\int_{a_2}^{y} f + \int_{x}^{x+h}\int_{a_2}^{y} f + \int_{a_1}^{x}\int_{y}^{y+k} f + \int_{x}^{x+h}\int_{y}^{y+k} f - \int_{a_1}^{x}\int_{a_2}^{y} f\right| \\ &= \left|\int_{x}^{x+h}\int_{a_2}^{y} f + \int_{a_1}^{x}\int_{y}^{y+k} f + \int_{x}^{x+h}\int_{y}^{y+k} f\right| \\ &\le h(y-a_2)M + (x-a_1)kM + hkM\end{aligned}$$

and the RHS goes to 0 as $(h,k) \to 0$. Using similar arguments, we can show that

$$\lim_{(h,k)\to(0,0)} |F(x-h,y+k) - F(x,y)| = 0$$
$$\lim_{(h,k)\to(0,0)} |F(x+h,y-k) - F(x,y)| = 0$$
$$\lim_{(h,k)\to(0,0)} |F(x-h,y-k) - F(x,y)| = 0$$

and hence F is a continuous function on R (this is a generalisation of the proof of the FTOC in one variable).

Next, we will show that F is $\mathscr{C}^1$ in the interior of the rectangle R, and to do this we will show that the partial derivatives D_1F and D_2F both exist and are continuous in the interior of R. Moreover, we will only do the proof for D_1F, as the proof for D_2F is very similar.

Let (x,y) be a point in the interior of R, i.e $a_1 < x < b_1$ and $a_2 < y < b_2$. For $t \in [a_1, b_1]$, define the function

$$g(t) = \int_{a_2}^{y} f(t,t_2)dt_2$$

Let us show that g is a continuous function on $[a_1,b_1]$. Let $\epsilon > 0$ be given. Since f is continuous on R and since R is compact, f is *uniformly continuous* on R. So there is some $\delta > 0$ such that for any $s,t \in [a_1,b_1]$ and $t_2 \in [a_2,b_2]$

$$|t-s| < \delta \implies |f(t,t_2) - f(s,t_2)| < \frac{\epsilon}{(y-a_2)}$$

So, if $|t-s| < \delta$ and $t,s \in [a_1,b_1]$, then we have

$$\begin{aligned}|g(t)-g(s)| &= \left|\int_{a_2}^{y} f(t,t_2) - f(s,t_2)dt_2\right| \\ &\leq \int_{a_2}^{y} |f(t,t_2) - f(s,t_2)|dt_2 \\ &\leq (y-a_2)\cdot\frac{\epsilon}{(y-a_2)} \\ &= \epsilon\end{aligned}$$

and this shows that g is continuous on $[a_1,b_1]$ (and infact, it is *uniformly continuous*). Now, by Fubini's Theorem, we know that for any $p \in [a_1,b_1]$,

$$F(p,y) = \int_{a_1}^{p}\left(\int_{a_2}^{y} f(t,t_2)dt_2\right)dt = \int_{a_1}^{p} g(t)dt$$

and hence by the Fundamental Theorem of Calculus in one variable, we get that

$$D_1F(x,y) = g(x) = \int_{a_2}^{y} f(x,t_2)dt_2$$

By a very similar argument, we can obtain

$$D_2F(x,y) = \int_{a_1}^{x} f(t_1,y)dt_1$$

So, both the partial derivatives D_1F and D_2F exist in the interior of R. It remains to prove the continuity of these. Let $(x,y) \in (a_1,b_1)\times(a_2,b_2)$ be fixed. Since f is continuous on R, it is uniformly continuous. Let $\epsilon > 0$ be given. There is some $\delta > 0$ such that for any $0 \le h < \delta$ and $t_2 \in [a_2, b_2]$,

$$|f(x+h, t_2) - f(x, t_2)| < \epsilon/(y - a_2)$$

Now, let (h, k) be such that $0 \le h < \delta$ and $k > 0$ is small enough. In that case, we have

$$\begin{aligned} |D_1F(x+h, y+k) - D_1F(x,y)| &= \left| \int_{a_2}^{y+k} f(x+h, t_2)dt_2 - \int_{a_2}^{y} f(x, t_2)dt_2 \right| \\ &= \left| \int_{a_2}^{y} f(x+h, t_2) - f(x, t_2)dt_2 + \int_{y}^{y+k} f(x+h, t_2)dt_2 \right| \\ &\le (y - a_2)\epsilon/(y - a_2) + kM \\ &= \epsilon + kM \end{aligned}$$

and the last term goes to ϵ as $(h, k) \to 0$. Since $\epsilon > 0$ is arbitrary, it must be true that

$$\lim_{(h,k)\to 0} |D_1F(x+h, y+k) - D_1F(x, y)| = 0$$

Using similar arguments, we can show that following limits (observe $h, k \ge 0$)

$$\begin{aligned} &\lim_{(h,k)\to 0} |D_1F(x-h, y+k) - D_1F(x, y)| = 0 \\ &\lim_{(h,k)\to 0} |D_1F(x+h, y-k) - D_1F(x, y)| = 0 \\ &\lim_{(h,k)\to 0} |D_1F(x+h, y-k) - D_1F(x, y)| = 0 \end{aligned}$$

and hence D_1F is continuous at (x, y). So, D_1F is continuous in the interior of R, and similarly D_2F is continuous in the interior of R. So, F is $\mathscr{C}^1$ in the interior of R.

(6). Let $R = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$ and f a $\mathscr{C}^1$ real valued function defined on an open set containing R. Let G be defined on $[a_2, b_2]$ by

$$G(y) = \int_{a_1}^{b_1} f(x, y)dx$$

Prove that

$$G'(y) = \int_{a_1}^{b_1} \frac{\partial f}{\partial y}(x, y)dx$$

Solution. We know that D_2F is continuous on R. So, by the Fundamental Theorem of Calculus in one variable, we know that for any $(x, y) \in [a_1, b_1] \times [a_2, b_2]$

$$f(x, y) = f(x, y) - f(x, a_2) + f(x, a_2) = \int_{a_2}^{y} D_2f(x, t_2)dt_2 + f(x, a_2)$$

So, we see that

$$\begin{aligned} G(y) &= \int_{a_1}^{b_1} f(x,y)dx \\ &= \int_{a_1}^{b_1} \left(\int_{a_2}^{y} D_2 f(x,t_2)dt_2 + f(x,a_2) \right) dx \\ &= \int_{a_1}^{b_1} \int_{a_2}^{y} D_2 f(x,t_2)dt_2 dx + \int_{a_1}^{b_1} f(x,a_2)dx \\ &= \int_{a_2}^{y} \int_{a_1}^{b_1} D_2 f(x,t_2)dx dt_2 + \int_{a_1}^{b_1} f(x,a_2)dx \qquad (**) \end{aligned}$$

where we used Fubini's Theorem to switch the order of integration in the last step. Now, for any $t \in [a_2, b_2]$, define the function

$$h(t) = \int_{a_1}^{b_1} D_2 f(x,t)dx$$

Let us show that h is a continuous function on $[a_2, b_2]$. The argument is very similar to that in problem **(5).** Since D_2 is continuous over R, it is *uniformly continuous*. Hence, if $\epsilon > 0$ is given, there is some $\delta > 0$ such that for $t, s \in [a_2, b_2]$,

$$|t - s| < \delta \implies |D_2 f(x,t) - D_2 f(x,s)| < \epsilon/(b_1 - a_1)$$

for any $x \in [a_1, b_1]$. So, it follows that if $t, s \in [a_1, b_1]$ and $|t - s| < \delta$, then

$$\begin{aligned} |h(t) - h(s)| &= \left| \int_{a_1}^{b_1} D_2 f(x,t) - D_2 f(x,s)dx \right| \\ &\le \int_{a_1}^{b_1} |D_2 f(x,t) - D_2 f(x,s)|dx \\ &\le (b_1 - a_1)\epsilon/(b_1 - a_1) \\ &= \epsilon \end{aligned}$$

showing that h is *uniformly continuous* over $[a_1, b_1]$. From equation $(**)$, we see that

$$G(y) = \int_{a_2}^{y} h(t_2)dt_2 + \int_{a_1}^{b_1} f(x,a_2)dx$$

Note that the second integral is independent of y. So, by the Fundamental Theorem of Calculus in one variable, we get

$$G'(y) = h(y) = \int_{a_1}^{b_1} D_2 f(x,y)dx$$

completing the proof.

(7). (Problem 3-33 in Spivak) If $f : [a,b] \times [c,d] \to \mathbb{R}$ is continuous and $D_2 f$ is continuous, define $F(x,y) = \int_a^x f(t,y)dt$.
(a) Find $D_1 F$ and $D_2 F$.

(b) If $G(x) = \int_a^{g(x)} f(t,x)dt$, find $G'(x)$ where $g : [c,d] \to [a,b]$ is a differentiable function.

Solution. Suppose $f : [a,b] \times [c,d] \to \mathbb{R}$ is continuous and $D_2 f$ is continuous. Let

$$F(x,y) = \int_a^x f(t,y)dt$$

(1) We find $D_1 F$ and $D_2 F$. First, by Leibniz' rule, we see that

$$D_2 F(x,y) = \int_a^x D_2 f(t,y)dt$$

Just as in problem **(5).**, this integral is continuous, i.e $D_2 F$ is continuous. Also, by the fundamental theorem of calculus, we have that

$$D_1 F(x,y) = f(x,y)$$

because for fixed y, the function $f(t,y)$ on $[a,b]$ is continuous. Clearly, $D_1 F$ is continuous, since f is continuous. Because both $D_1 F$ and $D_2 F$ exist and are continuous in the interior of the rectangle, it follows that F is $\mathscr{C}^1$ in the interior of the rectangle.

(2) Suppose

$$G(x) = \int_a^{g(x)} f(t,x)dt$$

for some differentiable function $g : [c,d] \to [a,b]$. So we can write

$$G = F \circ h$$

where $h : [c,d] \to \mathbb{R}^2$ is the map $h(x) = (g(x), x)$. Clearly, h is differentiable on (c,d), and hence G, being a composite of differentiable functions, is differentiable (note that F is $\mathscr{C}^1$). So, applying the chain rule, we see that

$$G'(x) = \begin{bmatrix} D_1 F(g(x),x) & D_2 F(g(x),x) \end{bmatrix} \begin{bmatrix} g'(x) \\ 1 \end{bmatrix} = g'(x) f(g(x),x) + \int_a^{g(x)} D_2 f(t,x)dt$$

(8). (Problem 3-36 in Spivak) **Cavalieri's principle:** Let A and B be Jordan-measurable subsets of $\mathbb{R}^3$. Let

$$A_c := \{(x,y) : (x,y,c) \in A\}$$

and define B_c similarly. Suppose each A_c and B_c are Jordan measurable and have the same area. Show that A and B have the same volume.

Solution. As we will see, this is a direct consequence of Fubini's Theorem. Since A, B are Jordan Measurable, they are bounded sets which have a well-defined volume. Let $R = [a_1,b_1] \times [a_2,b_2] \times [a_3,b_3]$ be a rectangle in $\mathbb{R}^3$ containing both A, B. Let χ_A and χ_B be the characteristic functions of A and B respectively. So, we know that

$$\int_R \chi_A$$

and

$$\int_R \chi_B$$

both exist and are respectively the volumes of A and B. By Fubini's Theorem, we know that

$$\int_R \chi_A = \int_{a_3}^{b_3} \mathcal{U}(c)dc \tag{†}$$

$$\int_R \chi_B = \int_{a_3}^{b_3} \mathcal{U}'(c)dc \tag{$*$}$$

where $\mathcal{U}, \mathcal{U}' : [a_3, b_3] \to \mathbb{R}$ are given by

$$\mathcal{U}(c) = \overline{\int_{[a_1,b_1]\times[a_2,b_2]}} \chi_A(x,y,c)dxdy = \overline{\int_{[a_1,b_1]\times[a_2,b_2]}} \chi_{A_c}(x,y)dxdy$$

$$\mathcal{U}'(c) = \overline{\int_{[a_1,b_1]\times[a_2,b_2]}} \chi_B(x,y,c)dxdy = \overline{\int_{[a_1,b_1]\times[a_2,b_2]}} \chi_{B_c}(x,y)dxdy$$

Now, we know that both the sets A_c and B_c are Jordan Measurable, and hence both the upper integrals in the above equations can be replaced by just integrals. Moreover, we know that A_c and B_c have the same area, and hence

$$\int_{[a_1,b_1]\times[a_2,b_2]} \chi_{A_c}(x,y)dxdy = \int_{[a_1,b_1]\times[a_2,b_2]} \chi_{B_c}(x,y)dxdy$$

and hence we see that $\mathcal{U}(c) = \mathcal{U}'(c)$ for every $c \in [a_3, b_3]$. So by the equations (†) and ($*$), we get that

$$\int_R \chi_A = \int_R \chi_B$$

and hence, A and B have equal volume. This completes the proof.