ASSIGNMENT-4

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(1). Here we compare two ways of defining absolutely integrable functions on $\mathbb{R}$.

Solution. Consider the open cover of $\mathbb{R}$ given by the intervals $(n-1, n+1)$ for $n \in \mathbb{Z}$. Let ψ be the function defined by

$$\psi(x) = \begin{cases} 0 & \text{, if } |x| \geq 3/4 \\ 1 & \text{, if } |x| \leq 1/4 \\ 2(3/4 - x) & \text{, if } 1/4 \leq x \leq 3/4 \\ 2(3/4 + x) & \text{, if } -3/4 \leq x \leq -1/4 \end{cases}$$

It is clear that supp$(\psi) = [-3/4, 3/4]$. If we define ψ_n for $n \in \mathbb{Z}$ by

$$\psi_n(x) = \psi(x - n)$$

then note that ψ_n is just translating the graph of ψ such that $(n, 0)$ is the new origin. So, we see that

$$\text{supp}(\psi_n) = [n - 3/4, n + 3/4] \subset (n-1, n+1)$$

We first show that ψ_n form a partition of unity with respect to the given cover. It is clear that each ψ_n has compact support contained inside $(n-1, n+1)$. Observe that if $x \in \mathbb{R}$, then only finitely many open intervals of the form $(n-1, n+1)$ intersect with the ball $B(x, 1/2)$; this is because n is ranging over the set $\mathbb{Z}$. So, this shows that only finitely many ψ_n are non-zero on this neighborhood of x. Now suppose $x \in \mathbb{Z}$. So the only open interval containing x is $(x-1, x+1)$. In that case, we have

$$\phi_x(x) = \phi(x - x) = \phi(0) = 1$$

Next, suppose $x \notin \mathbb{Z}$, so that $n - 1 < x < n$ for some $n \in \mathbb{Z}$. In this case, the only open intervals containing x are $(n-2, n)$ and $(n-1, n+1)$. The corresponding functions are ψ_{n-1} and ψ_n. So we must show that

$$\psi_{n-1}(x) + \psi_n(x) = 1$$

Now a couple of cases are possible.

(1) In the first case, we have $n - 1 < x \leq n - 3/4$. So, observe that

$$\psi_n(x) = \psi(x - n) = 0 \quad , \quad \text{as } x - n \leq -3/4$$

and

$$\psi_{n-1}(x) = \psi(x - (n-1)) = 1 \quad , \quad \text{as } 0 \leq x - (n-1) \leq 1/4$$

so in this case we see that $\psi_{n-1}(x) + \psi_n(x) = 1$.

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(2) In the next case, we have that $n - 3/4 \leq x \leq n - 1/4$. Observe that
$$\psi_n(x) = 2(3/4 + x - n) \quad , \quad \text{as } -3/4 \leq x - n \leq -1/4$$
and
$$\psi_{n-1}(x) = 2(3/4 - (x - (n-1))) \quad , \quad \text{as } 1/4 \leq x - (n-1) \leq 3/4$$
and hence in this case we have
$$\psi_n(x) + \psi_{n-1}(x) = 3/2 + 3/2 - 2 = 1$$
(3) In the final case, we have $n - 1/4 \leq x < n$. This case is symmetric to the case (1), and the proof is very similar.

So, it follows that $\{\psi_n\}$ indeed is a partition of unity with respect to this cover.

Next, let f be a real valued function on $\mathbb{R}$. We show that the following three statements are equivalent.

(1) $f|_{[-m,m]}$ is integrable for each $m \in \mathbb{N}$ and the increasing sequence
$$\int_{[-m,m]} |f(x)|dx$$
tends to a finite limit.

(2) $f|_{[m_1,m_2]}$ is integrable for each $m_1 < m_2 \in \mathbb{Z}$ and the set
$$\int_{[m_1,m_2]} |f(x)|dx$$
is bounded above.

(3) $f|_{[n-1,n+1]}$ is integrable for each $n \in \mathbb{Z}$ and the series
$$\sum_{n\in\mathbb{Z}} \int_{[n-1,n+1]} \psi_n(x)|f(x)|dx$$
is convergent.

Let us first show that (1) $\implies$ (2). Let $m_1 < m_2 \in \mathbb{Z}$, and let $m \in \mathbb{Z}$ such that $[m_1, m_2] \subsetneq [-m, m]$. Since f is integrable on $[-m, m]$, it is clear that f is also integrable on $[m_1, m_2]$ since $[m_1, m_2]$ is an acceptable set. Moreover, it is easy to see that
$$\int_{[m_1,m_2]} |f(x)|dx \leq \int_{[-m,m]} |f(x)|dx \leq \lim_{m\to\infty} \int_{[-m,m]} |f(x)|dx$$
and so the given set is bounded above.

Next, we prove that (2) $\implies$ (3). If $n \in \mathbb{Z}$ then clearly $n-1 < n+1$ and hence f is integrable on $[n-1, n+1]$. Now to show that the given series is convergent, it is enough to show that its partial sums are bounded because there are only positive terms involved. Now let $k \in \mathbb{Z}$, and consider the interval $[-k, k]$. Consider the intervals
$$[-k, -k+2], [-k+1, -k+3], ..., [k-3, k-1], [k-2, k]$$
whose union is $[-k, k]$. Now we see that
$$\sum_{j=-(k-1)}^{k-1} \int_{[j-1,j+1]} \psi_j(x)|f(x)|dx = \sum_{j=-(k-1)}^{k-1} \int_{[-k,k]} \psi_j(x)|f(x)|dx$$

where the above is true because supp$(\psi_j) \subset [j-1, j+1]$. Now, we see that

(†)

$$\sum_{j=-(k-1)}^{k-1} \int_{[-k,k]} \psi_j(x)|f(x)|dx = \int_{[-k,k]} \left(\sum_{j=-(k-1)}^{k-1} \psi_j(x) \right) |f(x)|dx \leq \int_{[-k,k]} |f(x)|dx$$

where we are using the fact that

$$\sum_{j=-(k-1)}^{k-1} \psi_j(x) \leq 1 \quad , \quad \text{for any } x \in [-k,k]$$

So, this shows that the partial sums of the given series are bounded above, and hence the series is convergent.

Finally, we show that (3) $\implies$ (1). So suppose the given series is convergent. It is enough to show that the sequence

$$\int_{[-m,m]} |f(x)|dx \quad , \quad m \in \mathbb{N}$$

is bounded above, since it is an increasing sequence. Consider the open intervals

$$(-m-1, -m+1), (-m, -m+2), ..., (m-2, m), (m-1, m+1)$$

All of these open intervals cover $[-m, m]$. Also, consider the corresponding functions

$$\psi_{-m}, \psi_{-m+1}, ..., \psi_{m-1}, \psi_m$$

Now let $x \in [-m, m]$. If x is an integer, then

$$\phi_x(x) = 1 \quad \text{and} \quad \psi_k(x) = 0 \text{ for } k \neq x, \; -m \leq k \leq m$$

and hence

$$\sum_{k=-m}^{m} \psi_k(x) = 1$$

If x is not an integer, then as we have shown before we have that

$$\psi_{[x]}(x) + \psi_{[x]+1}(x) = 1 \quad \text{and} \quad \psi_k(x) = 0 \text{ for } k \neq [x], [x]+1, \; -m \leq k \leq m$$

and again we see that

$$\sum_{k=-m}^{m} \psi_k(x) = 1$$

So in all cases, we see that

$$\begin{aligned}
\int_{[-m,m]} |f(x)|dx &= \int_{[-m,m]} \left(\sum_{k=-m}^{m} \psi_k(x) \right) |f(x)|dx \\
&= \sum_{k=-m}^{m} \int_{[-m,m]} \psi_k(x)|f(x)|dx \\
&\leq \sum_{k=-m}^{m} \int_{[k-1,k+1]} \psi_k(x)|f(x)|dx \\
&\leq \lim_{m\to\infty} \sum_{k=-m}^{m} \int_{[k-1,k+1]} \psi_k(x)|f(x)|dx \qquad (‡)
\end{aligned}$$

where in the last step we have used the fact that supp$(\psi_k) \subset [k-1, k+1]$ and also in the last step we have an inequality because $(-m-1, -m+1)$ and $(m-1, m+1)$ are not subsets of $[-m, m]$. So this shows that the increasing sequence in (1) is bounded, and hence it tends to a finite limit.

So we have shown the equivalence of conditions (1)-(3). Now suppose f is any function that satisfies any of these conditions, and hence it satisfies *all* of these conditions. Clearly, the limit

$$\lim_{m\to\infty} \sum_{n\le|m|} \int_{[n-1,n+1]} \psi_n(x) f(x) dx$$

exists because the given series is assumed to be absolutely convergent. Now, we can write

$$f = f_+ - f_-$$

where $f_+ = \max\{0, f\}$ and $f_- = \max\{0, -f\}$. Moreover, we can write

$$f_+ = \frac{|f| + f}{2} \quad \text{and } f_- = \frac{|f| - f}{2}$$

and the integrability of f and $|f|$ on intervals implies the integrability of f_+ and f_- on each interval of $\mathbb{R}$. So,

$$\int_{[-m,m]} f(x)dx = \int_{[-m,m]} f_+(x)dx - \int_{[-m,m]} f_-(x)dx$$

Since the limit

$$\lim_{m\to\infty} \int_{[-m,m]} |f(x)|dx$$

exists, it follows that both the limits

$$\lim_{m\to\infty} \int_{[-m,m]} f_+(x)dx \text{ and } \lim_{m\to\infty} \int_{[-m,m]} f_-(x)dx$$

exist and hence it follows that the limit

$$\lim_{m\to\infty} \int_{[-m,m]} f(x)dx$$

also exists. Finally, we will show that

$$\lim_{m\to\infty} \int_{[-m,m]} f(x)dx = \lim_{m\to\infty} \sum_{n\le|m|} \int_{[n-1,n+1]} \psi_n(x) f(x) dx$$

Let $\epsilon > 0$ be given. Then, there is some $M \in \mathbb{N}$ such that for all $|n| \geq M$, we have that

$$\int_{[n-1,n+1]} \psi_n(x)|f(x)|dx < \epsilon/2$$

and this is because of the convergence of the series given in condition (3). Now, let $m \in \mathbb{N}$ be any integer such that $m > M$, which means that $|m| > M$. Now as we did in proving the inequality (‡), consider the open intervals

$$(-m-1, -m+1), (-m, m+2), ..., (m-2, m), (m-1, m+1)$$

and the corresponding functions

$$\psi_{-m}, \psi_{-m+1}, ..., \psi_{m-1}, \psi_m$$

As we have proven above, we have that for any $x \in [-m, m]$

$$\sum_{n=-m}^{m} \psi_n(x) = 1$$

So, we see that

$$\begin{aligned}
\int_{[-m,m]} f(x)dx &= \int_{[-m,m]} \left(\sum_{n=-m}^{m} \psi_n(x) \right) f(x)dx \\
&= \sum_{n=-m}^{m} \int_{[-m,m]} \psi_n(x) f(x)dx \\
&= \int_{[-m,-m+1]} \psi_{-m}(x) f(x)dx + \sum_{n=-m+1}^{m-1} \int_{[n-1,n+1]} \psi_n(x) f(x)dx + \int_{[m-1,m]} \psi_m(x) f(x)dx
\end{aligned}$$

and again we have used the fact that supp$(\psi_n) \subset [n-1, n+1]$. The above implies that

$$\begin{aligned}
&\left| \int_{[-m,m]} f(x)dx - \sum_{n=-m+1}^{m-1} \int_{[n-1,n+1]} \psi_n(x) f(x)dx \right| \\
&= \left| \int_{[-m,-m+1]} \psi_{-m}(x) f(x)dx + \int_{[m-1,m]} \psi_m(x) f(x)dx \right| \\
&\leq \int_{[-m,-m+1]} \psi_{-m}(x) |f(x)|dx + \int_{[m-1,m]} \psi_m(x) |f(x)|dx \\
&\leq \int_{[-m-1,-m+1]} \psi_{-m}(x) |f(x)|dx + \int_{[m-1,m+1]} \psi_m(x) |f(x)|dx \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

Since $m > M$ was arbitrary, this shows that

$$\lim_{m \to \infty} \int_{[-m,m]} f(x)dx = \lim_{m \to \infty} \sum_{|n| \leq m} \int_{[n-1,n+1]} \psi_n(x) f(x)dx$$

completing our proof. ■

(2). Define the function $G_2 : \mathbb{R}^2 \to \mathbb{R}$ by $G_2(x, y) = \exp -x^2 - y^2$.

(a). First, we show that G_2 is absolutely integrable on $\mathbb{R}^2$. We note that G_2 is a non-negative continuous function on $\mathbb{R}^2$, and hence to prove that G_2 is absolutely integrable on $\mathbb{R}^2$, it is enough to show that the integral of G_2 on every rectangle in $\mathbb{R}^2$ is bounded above. So suppose there is a rectangle $[-L, L] \times [-L, L]$ in $\mathbb{R}^2$. Then, using **Fubini's Theorem** and using the fact that G_2 is a continuous function, we have

$$\int_{[-L,L]^2} e^{-x^2-y^2}dxdy = \int_{[-L,L]} e^{-y^2}\left(\int_{[-L,L]} e^{-x^2}dx\right)dy$$
$$= \left(\int_{[-L,L]} e^{-x^2}dx\right)\left(\int_{[-L,L]} e^{-y^2}dy\right)$$
$$= \left(\int_{[-L,L]} e^{-x^2}dx\right)^2$$

Next, we show that the function $G_1 : \mathbb{R} \to \mathbb{R}$ defined by $G_1(x) = e^{-x^2}$ is absolutely integrable on $\mathbb{R}$. Again, note that G_1 is a non-negative continuous function on $\mathbb{R}$, and hence it is enough to show that the integral of G_1 on every interval of $\mathbb{R}$ is bounded above. But this has been done in class (in particular, page 45 of the Lecture Notes). So, it follows that G_2 is integrable on $\mathbb{R}^2$. ■

(b). Couldn't do it. ■

(c). (Note: there was an error in the assignment; the range of θ is between $-\pi$ and π and the set should be $\{(x,0) \mid x \leq 0\}$. See below). Consider the map

$$\Phi : \{(r,\theta) \mid r > 0, -\pi < \theta < \pi\} \to \mathbb{R}^2 \setminus \{(x,0) \mid x \leq 0\}$$

given by

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$$

We show that Φ is a $\mathscr{C}^1$ diffeomorphism. It is enough to show that Φ is a one-one $\mathscr{C}^1$ map with invertible derivative at every point in the domain. It is clear that Φ is a one-one map, because we are restricting θ to the range $(-\pi,\pi)$. Moreover, it is $\mathscr{C}^1$, because each of its component functions are $\mathscr{C}^1$ at each point in the domain (and infact they are $\mathscr{C}^\infty$). Finally, the Jacobian at any point (r,θ) is given by

$$J\Phi(r,\theta) = \det\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

and hence the derivative at every point in the domain is invertible. So, this shows that Φ is a $\mathscr{C}^1$ diffeomorphism. ■

(3). Here I will be solving problems 7,8,9 of Rudin's Chapter 6.

Problem 7. Let $f : (0,1] \to \mathbb{R}$ be a real function such that f is Riemann Integrable on $[c,1]$ for *every* $c > 0$. We define

$$\int_0^1 f(x)dx = \lim_{c\to 0}\int_c^1 f(x)dx$$

if the limit on the RHS exists and is finite.
(a) Suppose f is Riemann Integrable on $[0,1]$. Also, suppose $|f(x)| \leq M$ for $x \in [0,1]$ (the Riemann Integral is only defined for bounded functions). We will show that this definition agrees with the old one, i.e we will show that the limit

$$\lim_{c\to 0}\int_c^1 f(x)dx$$

exists and is equal to $\int_0^1 f(x)dx$. Let $c > 0$. Then, we have

$$\int_0^1 f(x)dx = \int_0^c f(x)dx + \int_c^1 f(x)dx$$

which implies that

$$\left|\int_0^1 f(x)dx - \int_c^1 f(x)dx\right| = \left|\int_0^c f(x)dx\right| \leq \int_0^c |f(x)|dx \leq Mc$$

The above inequality implies that as $c \to 0$,

$$\left|\int_0^1 f(x)dx - \int_c^1 f(x)dx\right| \to 0$$

and this proves the claim.

(b) We construct a function f for which the above limit exists, but it fails to exist for $|f|$ in place of f. The idea will involve an *alternating series*.

Define a function $f : (0, 1] \to \mathbb{R}$ as follows.

$$f(x) = (-1)^n(n+1) \quad , \quad x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$$

i.e we have defined f piecewise. It is clear that for every $c > 0$, f is integrable on $[c, 1]$ because it is a sum of piecewise constant functions. More explicitly, suppose $c > 0$ such that

$$\frac{1}{n+1} \leq c \leq \frac{1}{n}$$

for some $n \in \mathbb{N}$. Then by summing areas of rectangles, we see that

$$\int_c^1 f(x)dx = \left(\frac{1}{n} - c\right)(-1)^n(n+1) + \sum_{k=1}^{n-1} \frac{(-1)^k(k+1)}{k(k+1)}$$

The second series above is just

$$\sum_{k=1}^{n-1} \frac{(-1)^k}{k}$$

which is a partial sum of the alternating harmonic series. Now as $c \to 0$, we see that $n \to \infty$, and hence

$$\left(\frac{1}{n} - c\right)(-1)^n(n+1) \leq \left(\frac{1}{n} - \frac{1}{n+1}\right)(-1)^n(n+1) \to 0$$

So we see that

$$\lim_{c\to 0} \int_c^1 f(x)dx = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

and hence this limit exists. Now, we again by summing areas of rectangles, we can see that for $c > 0$ with

$$\frac{1}{n+1} \leq c \leq \frac{1}{n}$$

we have

$$\int_c^1 |f(x)|dx = \int_c^1 f(x)dx = \left(\frac{1}{n} - c\right)(n+1) + \sum_{k=1}^{n-1} \frac{(k+1)}{k(k+1)}$$

As before, as $c \to 0$, we have $n \to \infty$ and hence

$$\left(\frac{1}{n} - c\right)(n+1) \leq \left(\frac{1}{n} - \frac{1}{n+1}\right)(n+1) \to 0$$

But, the partial sums
$$\sum_{k=1}^{n-1} \frac{(k+1)}{k(k+1)} = \sum_{k=1}^{n-1} \frac{1}{k}$$
diverge by the divergence of the harmonic series. Hence, the limit
$$\lim_{c\to 0} \int_c^1 |f(x)|dx = \infty$$
and hence it does not exist. ■

Problem 8. Suppose $f(x) \geq 0$ and f decreases monotonically on $[1, \infty)$. We show that
$$\int_1^\infty f(x)dx$$
converges if and only if
$$\sum_{n=1}^{\infty} f(n)$$
converges. First, suppose the given integral converges. Let $n > 1$ be any natural number. Consider the partition
$$[1, 2] \cup [2, 3] \cup ... \cup [n-1, n]$$
of the interval $[1, n]$. Since f is decreasing, the lower Riemann sum of f over this partition is
$$f(2) + f(3) + ... + f(n) = \sum_{k=2}^{n} f(k)$$
and hence hence we see that
$$\sum_{k=2}^{n} f(k) \leq \int_1^n f(x)dx \leq \int_1^\infty f(x)dx$$
So, the partial sums of the series are all bounded above, and since the series consists of only positive terms, we see that the series is also convergent. Conversely, suppose the given series is convergent. Again, consider the interval $[1, n]$ for $n > 1$ and the same partition. Since f is decreasing, the upper Riemann sum of f with respect to this partition is
$$U(P, f) = f(1) + f(2) + ... + f(n-1) = \sum_{k=1}^{n-1} f(k)$$
and hence
$$\int_1^n f(x)dx \leq U(P, f) = \sum_{k=1}^{n-1} f(k) \leq \sum_{k=1}^{\infty} f(k)$$
Because f is a positive function and since the integral of f over $[1, n]$ is bounded above for every $n > 1$, we see that the integral of f over $[1, b]$ is bounded above for every $b > 1$, and hence
$$\int_1^n f(x)dx < \infty$$
and this completes the proof. ■

Here are two facts that I will prove before solving the next problem.

Proposition 0.1 (**Cauchy Criterion**). *Let $a \in \mathbb{R}$ be fixed, and let $f : [a, \infty) \to \mathbb{R}$ be a function such that f is integrable on every interval $[a, b]$ for $b > a$. Then, the integral*

$$\int_a^\infty f(x)dx$$

converges if and only if for every $\epsilon > 0$ there is an $M \geq a$ such that for all $B \geq A \geq M$ we have

$$\left|\int_A^B f(x)dx\right| < \epsilon$$

Proof. First, suppose that the given improper integral converges to $L \in \mathbb{R}$ and let $\epsilon > 0$ be given. Then, there is some $M \geq a$ such that for all $A \geq M$ we have

$$\left|\int_a^A f(x)dx - L\right| < \epsilon/2$$

Then, for any $B \geq A \geq M$ we have

$$\begin{aligned}\left|\int_A^B f(x)dx\right| &= \left|\int_a^B f(x)dx - \int_a^A f(x)dx\right| \\ &= \left|\int_a^B f(x)dx - L + L - \int_a^A f(x)dx\right| \\ &\leq \left|\int_a^B f(x)dx - L\right| + \left|\int_a^A f(x)dx - L\right| \\ &< \epsilon\end{aligned}$$

Conversely, suppose the given Cauchy Criterion holds. For natural numbers $n \geq a$, let

$$a_n = \int_a^n f(x)dx$$

Let $\epsilon > 0$ and let $M \geq 0$ be such that for all natural numbers $n \geq m \geq M$, we have

$$|a_n - a_m| = \left|\int_m^n f(x)dx\right| < \epsilon$$

and hence $\{a_n\}$ is a Cauchy sequence, and so $\{a_n\}$ has a limit L. Again, let $\epsilon > 0$ be given and choose $M \geq a$ such that $|a_n - L| \leq \epsilon/2$ and

$$\left|\int_A^B f(x)dx\right| < \epsilon/2$$

for all $n, A, B \geq M$, $n \in \mathbb{N}$. Now, let $A \geq M + 1$, and hence $[A] \geq M$. For such A, we have

$$\begin{aligned}\left|\int_a^A f(x)dx - L\right| &= \left|\int_a^{[A]} f(x)dx - L + \int_A^{[A]} f(x)dx\right| \\ &\leq \left|\int_a^{[A]} f(x)dx - L\right| + \left|\int_A^{[A]} f(x)dx\right| \\ &< \epsilon\end{aligned}$$

and hence the given improper integral converges to L. This completes the proof. ∎

Proposition 0.2. *Let $f : [a, \infty) \to \mathbb{R}$ be a function such that f is integrable on every $[a, b]$, for $b > a$. Suppose the integral*

$$\int_a^\infty |f(x)|dx$$

converges. Then, the integral

$$\int_a^\infty f(x)dx$$

also converges. Hence, absolute convergence of improper integrals implies convergence.

Proof. Suppose the

$$\int_a^\infty |f(x)|dx$$

is convergent, and let $\epsilon > 0$ be given. So by the **Cauchy Criterion** 0.1 there is some $M \geq a$ such that for all $B \geq A \geq M$,

$$\int_A^B |f(x)|dx < \epsilon$$

So for such A, B we have

$$\left| \int_A^B f(x)dx \right| \leq \int_A^B |f(x)|dx < \epsilon$$

and hence again by the **Cauchy Criterion** 0.1 the integral

$$\int_a^\infty f(x)dx$$

is convergent. ■

Problem 9. We will prove an integration by parts formula for improper integrals. Let $a \in \mathbb{R}$ be a fixed number, and let $f, g : [a, \infty) \to \mathbb{R}$ be continuously differentiable functions on $[a, \infty)$ such that

$$\lim_{x \to \infty} f(x)g(x) = M$$

for some $M \in \mathbb{R}$ and the integral

$$\int_a^\infty f(x)g'(x)dx$$

converges. Then we show that the integral

$$\int_a^\infty f'(x)g(x)dx$$

converges and

$$\int_a^\infty f'(x)g(x)dx = M - f(a)g(a) - \int_a^\infty f(x)g'(x)dx$$

So let's prove this. Let $b > a$ be fixed. Applying integration by parts on the interval $[a, b]$, we see that

$$\int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx$$

The limit as $b \to \infty$ exists on the RHS by our hypothesis, and hence

$$\int_a^\infty f'(x)g(x)dx = M - f(a)g(a) - \int_a^\infty f(x)g'(x)dx$$

Let us apply this to a specific case. Consider the interval $[0, \infty)$. On this interval let

$$f(x) = \sin x$$
$$g(x) = \frac{1}{1+x}$$

so that both f and g are continuously differentiable on $[0, \infty)$. First, observe that

$$\int_0^\infty \frac{|\sin x|}{(1+x)^2}dx \leq \int_0^\infty \frac{1}{(1+x)^2}dx$$

Now, we know that the function

$$x \mapsto \frac{1}{(1+x)^2}$$

is a positive decreasing function on $[0, \infty)$. Using the result in **Problem 8**. we see that

$$\sum_{n=0}^{\infty} \frac{1}{(1+n)^2} \text{ converges } \iff \int_0^\infty \frac{1}{(1+x)^2}dx \text{ converges}$$

and we know that the series on the LHS above converges. So, we see that

$$\int_0^\infty \frac{|\sin x|}{(1+x)^2}dx < \infty$$

and hence by **Proposition** 0.2, we see that the integral

$$\int_0^\infty \frac{\sin x}{(1+x)^2}dx$$

converges. We have just shown that the integral

$$\int_0^\infty f(x)g'(x)dx = -\int_0^\infty \frac{\sin x}{(1+x)^2}dx$$

is convergent. Now, observe that for any $b > 0$,

$$f(b)g(b) = \frac{\sin b}{(1+b)^2} \to 0 \text{ as } b \to \infty$$

Applying the integration by parts formula we proved, we have

$$\int_0^\infty f'(x)g(x)dx = \int_0^\infty \frac{\cos x}{1+x}dx = 0 - \frac{\sin 0}{1+0} - \int_0^\infty f(x)g'(x)dx = \int_0^\infty \frac{\sin x}{(1+x)^2}dx$$

and hence

$$\int_0^\infty \frac{\cos x}{1+x}dx = \int_0^\infty \frac{\sin x}{(1+x)^2}dx$$

Now, we will show the the integral on the LHS above is not absolutely convergent, i.e

$$\int_0^\infty \frac{|\cos x|}{1+x}dx = \infty$$

Note that on any interval of the form $[2\pi k, 2\pi k+\pi/4]$ for $k \geq 0$, the function cos x is positive and is bounded below by cos$(\pi/4) = 1/\sqrt{2}$. Now we immediately see that

$$\begin{aligned}\int_0^\infty \frac{|\cos x|}{1+x}dx &\geq \sum_{k=0}^\infty \int_{2\pi k}^{2\pi k+\pi/4} \frac{1}{\sqrt{2}(1+x)}dx \\ &\geq \frac{1}{\sqrt{2}}\sum_{k=0}^\infty \int_{2\pi k}^{2\pi k+\pi/4} \frac{1}{1+2\pi k+\pi/4}dx \\ &= \frac{\pi}{4\sqrt{2}}\sum_{k=0}^\infty \frac{1}{1+2\pi k+\pi/4}\end{aligned}$$

The last series is of the form

$$c\sum_{k=0}^\infty \frac{1}{ak+b}$$

where $a, b > 1$ and $c > 0$. Clearly, this series diverges by comparison with the harmonic series. Hence, the given integral does *not* converge absolutely. ■

(4). This is problem 3-37 of Spivak.

Solution. **(a)** Suppose $f : (0,1) \to \mathbb{R}$ is a non-negative continuous function. We will show that $\int_{(0,1)} f$ exists if and only if lim$_{\epsilon\to 0}\, 0 \int_\epsilon^{1-\epsilon} f$ exists. Since $f \geq 0$, the function

$$g(\epsilon) = \int_\epsilon^{1-\epsilon} f$$

is increasing as $\epsilon \to 0$. So,

$$\lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} f$$

exists if and only if $\int_\epsilon^{1-\epsilon} f$ is bounded above. Let $\{R_i\}_{i\in\mathbb{N}}$ be a family of rectangles contained in $(0,1)$ such that IntR_i cover $(0,1)$ (that such a cover exists is proven in the Lecture Notes). Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a partition of unity subordinate to this cover, such that supp$(\varphi_i) \subset$ IntR_i for each $i \in \mathbb{N}$ (existence of this is mentioned in the Lecture Notes page 48).

Let $\epsilon > 0$. Because $[\epsilon, 1-\epsilon]$ is a compact set, only finitely many of the φ_i are not 0 on $[\epsilon, 1-\epsilon]$. So, we see that

$$\int_\epsilon^{1-\epsilon} f = \int_\epsilon^{1-\epsilon}\sum_{i\in\mathbb{N}}\varphi_i\cdot f = \sum_{i\in\mathbb{N}}\int_\epsilon^{1-\epsilon}\varphi_i\cdot f \leq \sum_{i\in\mathbb{N}}\int_{(0,1)}\varphi_i\cdot f$$

So, the above implies that $\int_\epsilon^{1-\epsilon} f$ is bounded above for $\epsilon > 0$, if $\int_{(0,1)} f$ exists. Hence, if the integral exists, then the given limit exists.

Conversely, suppose the given limit exists. Consider $\varphi_1, ..., \varphi_n$ for some $n > 0$. Since each function is compactly supported and there are finitely many of them, there is some $\epsilon > 0$ such that all of $\varphi_1, ..., \varphi_n$ are zero outside of $[\epsilon, 1-\epsilon]$. In that case, we have that

$$\sum_{i=1}^n\int_{(0,1)}\varphi_i\cdot f = \sum_{i=1}^n\int_\epsilon^{1-\epsilon}\varphi_i\cdot f = \int_\epsilon^{1-\epsilon}\sum_{i=1}^n\varphi_i\cdot f \leq \int_\epsilon^{1-\epsilon} f \leq \lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} f$$

and hence the partial sums of the series $\sum_{i\in\mathbb{N}} \int_{(0,1)} \varphi_i \cdot f$ are bounded above, i.e the given series converges. This implies that

$$\int_{(0,1)} f$$

exists and this completes the proof.
(b) Couldn't do it. ■

(5). In this problem we will show that any $\mathscr{C}^1$ curve γ is rectifiable and that

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)|dt$$

Solution. Let $\gamma : [a,b] \to \mathbb{R}^n$ by any $\mathscr{C}^1$ curve. First, suppose $P := a = x_0 < x_1 < \dots < x_k = b$ is any partition on $[a,b]$. Then by the **Fundamental Theorem of Calculus**, for any $1 \leq i \leq k$ we see that

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t)dt \right| \leq \int_{x_{i-1}}^{x_i} |\gamma'(t)|dt$$

Summing over all i, we get

$$\Lambda(P,\gamma) \leq \int_a^b |\gamma'(t)|dt$$

Because P was an arbitrary partition of $[a,b]$, we get

$$\text{length}(\gamma) \leq \int_a^b |\gamma'(t)|dt$$

Next we show that reverse inequality. Let $\epsilon > 0$ be given. Because γ' is continuous on $[a,b]$, it is uniformly continuous and hence there is some $\delta > 0$ such that

$$|\gamma'(s) - \gamma'(t)| < \epsilon$$

for all $s,t \in [a,b]$ with $|s-t| < \delta$. Now let

$$P := a = x_0 < x_1 < \dots < x_k = b$$

be any partition of $[a,b]$ such that $x_i - x_{i-1} < \delta$ for each $1 \leq i \leq k$. If $t \in [x_{i-1}, x_i]$ then we immediately see that

$$|\gamma'(t)| \leq |\gamma'(x_i)| + \epsilon$$

So, we have that

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} |\gamma'(t)|dt &\leq |\gamma'(x_i)|[x_i - x_{i-1}] + \epsilon[x_i - x_{i-1}] \\
&= \left| \int_{x_{i-1}}^{x_i} \gamma'(x_i)dt \right| + \epsilon[x_i - x_{i-1}] \\
&= \left| \int_{x_{i-1}}^{x_i} (\gamma'(t) + \gamma'(x_i) - \gamma'(t))dt \right| + \epsilon[x_i - x_{i-1}] \\
&\leq \left| \int_{x_{i-1}}^{x_i} \gamma'(t)dt \right| + \left| \int_{x_{i-1}}^{x_i} (\gamma'(x_i) - \gamma'(t))dt \right| + \epsilon[x_i - x_{i-1}] \\
&\leq |\gamma(x_i) - \gamma(x_{i-1})| + 2\epsilon[x_i - x_{i-1}]
\end{aligned}$$

where again in the last step we used the **Fundamental Theorem of Calculus** and the inequality given by the uniform continuity of γ'. Summing over all i, we see that

$$\int_a^b |\gamma'(t)|dt \leq \Lambda(P,\gamma) + 2\epsilon(b-1) \leq \text{length}(\gamma) + 2\epsilon(b-a)$$

Since ϵ is arbitrary, we get that

$$\int_a^b |\gamma'(t)|dt \leq \text{length}(\gamma)$$

and hence our proof is complete. ■

Proposition 0.3. *Let $[a,b],[c,d]$ be any intervals in $\mathbb{R}$. Let $\phi : [a,b] \to [c,d]$ be a continuous bijection. Then ϕ must be monotonic. Since it is a bijection, it implies that ϕ is strictly monotonic.*

Proof. It is enough to show that ϕ is monotonic. For the sake of contradiction, suppose it is not. So, there are $a_1 < a_2 < a_3$ in $[a,b]$ such that

$$\phi(a_1) < \phi(a_2) > \phi(a_3)$$

Let $z \in (\phi(a_1),\phi(a_2)) \cap (\phi(a_3),\phi(a_2))$. By the intermediate value theorem, we see that there are $x_1 \in (a_1,a_2)$ and $x_2 \in (a_2,a_3)$ such that

$$\phi(x_1) = \phi(x_2) = z$$

contradicting the fact that ϕ is a bijection. The other case when

$$\phi(a_1) > \phi(a_2) < \phi(a_3)$$

is handled similarly. This completes the proof. ■

(6). Here I will do problem 19 of Rudin's Chapter 6.

Solution. Let γ_1 be a curve in $\mathbb{R}^k$ defined on $[a,b]$; let ϕ be a continuous one-one mapping of $[c,d]$ onto $[a,b]$ such that $\phi(c) = a$. We define $\gamma_2(s) = \gamma_1(\phi(s))$ for $s \in [c,d]$.

First, suppose γ_1 is one-one. Then

$$\begin{aligned}\gamma_2(s) = \gamma_2(t) &\implies \gamma_1(\phi(s)) = \gamma_1(\phi(t))\\ &\implies \phi(s) = \phi(t)\\ &\implies s = t\end{aligned}$$

so that γ_2 is also one-one. Next, if γ_2 is one-one, then we have

$$\begin{aligned}\gamma_1(s) = \gamma_1(t) &\implies \gamma_1(\phi\circ\phi^{-1}(s)) = \gamma_1(\phi\circ\phi^{-1}(t))\\ &\implies \gamma_2(\phi^{-1}(s)) = \gamma_2(\phi^{-1}(t))\\ &\implies \phi^{-1}(s) = \phi^{-1}(t)\\ &\implies s = t\end{aligned}$$

and hence γ_1 is also one-one. This shows that γ_1 is an arc if and only if γ_2 is also an arc.

Because ϕ is a continuous bijection, we see by **Proposition** 0.3 that $\phi(d) = b$ because we already know that $\phi(c) = a$. So this shows that γ_1 is a closed curve if and only if γ_2 is a closed curve.

Finally, we show that γ_1 is rectifiable if and only if γ_2 is rectifiable. Because ϕ is a continuous bijection, **Proposition** 0.3 implies that ϕ is strictly monotonic, and because $\phi(c) = a$, we see that ϕ is monotonic increasing. So, ϕ gives a one-one correspondence between partitions of $[c, d]$ and partitions of $[a, b]$; if

$$P := c = t_0 < t_1 < ... < t_k = d$$

is a partition of $[c, d]$, then

$$P' := a = \phi(t_0) < \phi(t_1) < ... < \phi(t_k) = b$$

is a partition of $[a, b]$. Also, if P and P' are such corresponding partitions, then note that

$$\Lambda(P, \gamma_2) = \sum_{i=1}^{k} |\gamma_2(t_i) - \gamma_1(t_{i-1})| = \sum_{i=1}^{k} |\gamma_1(\phi(t_i)) - \gamma_1(\phi(t_{i-1}))| = \Lambda(P', \gamma_1)$$

This shows that γ_2 is rectifiable if and only if γ_1 is rectifiable; the same also shows that γ_1 and γ_2 have equal length. ■

(7). The same problem as **(6).**, except we assume that all parametrisations are $\mathscr{C}^1$ and we use the change of variables formula.

Solution. Let γ_1, γ_2 and ϕ be as in the previous problem, and suppose all these are $\mathscr{C}^1$ mappings. We have already shown that γ_2 is rectifiable if and only if γ_1 is rectifiable. By problem **(5).**, we know that

$$\text{length}(\gamma_1) = \int_a^b |\gamma_1'(t)| dt$$

$$\text{length}(\gamma_2) = \int_c^d |\gamma_2'(t)| dt$$

Now, we know that $\gamma_2 = \gamma_1 \circ \phi$, and hence for any $t \in [c, d]$

$$\gamma_2'(t) = \gamma_1'(\phi(t))\phi'(t)$$

and hence

$$|\gamma_2'(t)| = |\gamma_1'(\phi(t))||\phi'(t)|$$

Moreover, as we showed in the previous problem, we know that ϕ is an increasing function, i.e $\phi'(t) \geq 0$ for every $t \in [c, d]$. So,

$$|\gamma_2'(t)| = |\gamma_1'(\phi(t))||\phi'(t)| = |\gamma_1'(\phi(t))|\phi'(t)$$

for every $t \in [c, d]$. So by change of variables in one variable, we see that

$$\text{length}(\gamma_2) = \int_c^d |\gamma_2'(t)| dt = \int_c^d |\gamma_1'(\phi(t))|\phi'(t) dt = \int_a^b |\gamma_1'(t)| dt = \text{length}(\gamma_1)$$

and hence this proves that they have the same length. ■