CALCULUS

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These are my notes for the course **CALCULUS** which I took in my third semester. The notes are mostly self-contained. The main reference book I used was *Calculus on Manifolds* by *M.Spivak*. Throughout the document, the symbol ■ will mean QED.

Contents

1. Recap of the Riemann Integral in Dimension 1

In this section, we will revise some statements regarding the Riemann integral in one dimension. In subsequent sections, the goal will be to prove analogues in higher dimensions.

Lemma 1.1. *Any two partitions of an interval have a common refinement.*

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Proof. Let $I = [a, b]$ be our interval, and let $P = a = a_0 < a_1 < ... < a_n = b$, $P' = a = a_0' < ... < a_m' = b$ be two partitions. Consider the set $Q = \{a_0, ..., a_n\} \cup \{a_0', ..., a_m'\}$ and suppose $Q = \{a = b_0, ..., b_r = b\}$ be written in non-decreasing order. Consider the parition

$$P^* = a = b_0 < ... < b_r = b$$

and we claim that P^* is a common refinement of P and P'. But this is easy to see, because all points of P and P' are present in P^*, proving the claim. ■

Our next lemma is regarding the upper and lower sums of a partition and its refinements.

Lemma 1.2. *Suppose P is a partition of $[a, b]$, and let Q be a refinement. Then,*

$$m(b-a) \leq L(P, f) \leq L(Q, f) \leq U(Q, f) \leq U(P, f) \leq M(b-a)$$

where m, M are respectively the infimum and supremum of f over $[a, b]$ (it is assumed that f is a bounded function).

Proof. Without loss of generality, we can assume that Q has exactly one more point than P (otherwise we can repeat the argument one by one for every new point). So suppose $Q := a = a_0 < a_1 < ... < a_n = b$ and let $Q := a = a_0 < a_1' < a_1 < ... < a_n = b$ (the extra point is a_1'). Now,

$$U(P, f) = M_{I_1}(a_1 - a_0) + \sum_{i=2}^{n} M_{I_i}(a_i - a_{i-1})$$

where $I_i = [a_{i-1}, a_i]$ And similarly we have

$$U(Q, f) = M_{[a_0, a_1']}(a_1' - a_0) + M_{[a_1', a_1]}(a_1 - a_1') + \sum_{i=2}^{n} M_{I_i}(a_i - a_{i-1})$$

And hence we have

$$U(P, f) - U(Q, f) = M_{I_1}(a_1 - a_0) - M_{[a_0, a_1']}(a_1' - a_0) - M_{[a_1', a_1]}(a_1 - a_1') \geq 0$$

and hence we are done. The proof for lower sums is similar. ■

There is a *Cauchy-Criterion* for Riemann integrability, which is the following.

Theorem 1.3. *f is Riemann-integrable over $[a, b]$ if and only if for every $\epsilon > 0$, there is a partition P of $[a, b]$ for which*

$$U(P, f) - L(P, f) < \epsilon$$

Another important theorem we have is the integrability of continuous functions.

Theorem 1.4. *If f is continuous on $[a, b]$, then f is integrable on $[a, b]$ as well.*

The proof uses the important *uniform continuity* of f on $[a, b]$, which we state as a theorem.

Theorem 1.5. *Let X, Y be metric spaces where X is compact, and suppose $f : X \to Y$ be continuous. Then, f is uniformly continuous.*

Proposition 1.6. *Let I be a closed and bounded interval. If f is continuous at all except finitely many points of I, then f is integrable on I.*

Proposition 1.7. *Suppose f is Riemann-Integrable on $[a, b]$ such that $m \leq f \leq M$. Let $\phi : [m, M] \to \mathbb{R}$ be a continuous map. Then, $\phi \circ f : [a, b] \to \mathbb{R}$ is also integrable.*

2. Integration in Higher Dimensions

We first begin with the usual notion of integration over *rectangles* in $\mathbb{R}^n$. In the subsequent sections, integration over a larger class of sets will be considered.

Definition 2.1. Let $R = [a_1, b_1] \times ... \times [a_n, b_n]$ be a rectangle in $\mathbb{R}^n$. A *partition* of $\mathbb{R}$ is a collection $(P_1, P_2, ..., P_n)$ of partitions where P_i is a partition of the interval $[a_i, b_i]$ in $\mathbb{R}$. Suppose

$$\begin{aligned} P_1 &:= a_1 = t_{10} < t_{11} < ... < t_{1m_1} = b_1 \\ P_2 &:= a_2 = t_{20} < t_{21} < ... < t_{2m_2} = b_2 \\ &... \\ P_n &:= a_n = t_{n0} < t_{n1} < ... < t_{nm_n} = b_n \end{aligned}$$

be the partitions. Then, any rectangle of the form

$$[t_{1(j_1-1),t_{1j_1}}] \times [t_{2(j_2-1),t_{2j_2}}] \times ... \times [t_{n(j_n-1),t_{nj_n}}]$$

is called a *subrectangle* of R. Observe that the total number of subrectangles in this partition will be $m_1...m_n$.

Definition 2.2. Let R be a rectangle in $\mathbb{R}^n$ and let $P = (P_1, ..., P_n)$ be a partition of R as given in the above definition. Let $f : R \to \mathbb{R}$ be a bounded function. For every sub-rectangle S of R, let $v(S)$ denote its n-dimensional volume, and define

$$\begin{aligned} m_S(f) &:= \text{inf}\{f(x) : x \in S\} \\ M_S(f) &:= \text{sup}\{f(x) : x \in S\} \end{aligned}$$

Next, we define

$$\begin{aligned} U(P, f) &= \sum_S M_S(f)v(S) \\ L(P, f) &= \sum_S m_S(f)v(S) \end{aligned}$$

which are the usual upper and lower sums of f with respect to P.

It is clear that $L(P, f) \leq U(P, f)$. Some properties of upper and lower sums are given below, and the proofs are very similar to that of the one dimensional case.

Lemma 2.1. *Suppose the partition P' is a refinement of the partition P. Then,*

$$L(P, f) \leq L(P', f) \text{ and } U(P', f) \leq U(P, f)$$

Corollary 2.1.1. *If P and P' are any two partitions of R, then*

$$L(P', f) \leq U(P, f)$$

Definition 2.3. For a bounded function $f : R \to \mathbb{R}$, we define

$$\begin{aligned} \overline{\int}_R f &= \inf_P U(P, f) \\ \underline{\int}_R f &= \sup_P L(P, f) \end{aligned}$$

and f is said to be *integrable* if the above two numbers are equal, and this is denoted by

$$\int_R f = \int_R f(x^1, x^2, ..., x^n)dx^1...dx^n$$

As usual, the Cauchy-criterion for integrability still holds and is very easy to prove. Here are two examples.

Example 2.1. Let $f : R \to \mathbb{R}$ be a constant function, i.e $f(x) = c$ for all $x \in R$. It is easy to see that f is integrable over R, and that

$$\int_R f = cv(R)$$

Example 2.2. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x, y) = \chi_{\mathbb{Q}}(x)$$

where $\chi_{\mathbb{Q}}$ is the characteristic functions of the rationals. It is easy to see that f is not integrable over the given rectangle.

2.1. **Measure and Content.** Note that rectangles are not the only interesting sets in $\mathbb{R}^n$. There are other connected sets as well. The situation in $\mathbb{R}$ is not that difficult, because the only connected subsets of $\mathbb{R}$ are intervals.

Definition 2.4. A subset A of $\mathbb{R}^n$ has *measure* 0 if for every $\epsilon > 0$, there is a cover $\bigcup_{i\in\mathbb{N}} R_i$ of A where each R_i is a closed (or open) rectangle in $\mathbb{R}^n$ such that

$$\sum_{i=1}^{\infty} v(R_i) < \epsilon$$

Here, $v(R_i)$ is the n-dimensional volume of the closed (or open rectangle).

Remark 2.1.1. The fact that either open or closed rectangles can be used is powerful and is just a fact about the structure of rectangles in $\mathbb{R}^n$.

Definition 2.5. A subset A of $\mathbb{R}^n$ has *content* 0 if for every $\epsilon > 0$, there are *finitely many* closed (or open) rectangles $R_1, ..., R_k$ such that $A \subset R_1 \cup ... \cup R_k$ and

$$\sum_{i=1}^{k} v(R_i) < \epsilon$$

It is clear that a set with measure 0 also has content 0.

Proposition 2.2. *Any countable subset of* $\mathbb{R}^n$ *has measure zero.*

Proof. Enumerate the points as $\{a_1, ..., a_n, ...\}$. Take an open rectangle around a_n of volume less than $\epsilon/2^n$. The claim follows. ■

Proposition 2.3. *If* $A = \bigcup_{i=1}^{\infty} A_i$ *and each* A_i *has measure* 0*, then* A *has measure* 0.

Proof. Since A_i has measure 0 for each i, there are closed (or open rectangles) $U_{i,k}$ such that

$$A_i \subset \bigcup_{k=1}^{\infty} U_{i,k}$$

and

$$\sum_{k=1}^{\infty} v(U_{i,k}) < \epsilon/2^i$$

The collection of all rectangles $\{U_{i,k} : i, k \in \mathbb{N}\}$ is countable, and

$$\sum_{i,k\in\mathbb{N}} v(U_{i,k}) < \sum_{i=1}^{\infty} \epsilon/2^i$$

and hence A has measure 0. ∎

Proposition 2.4. *If A is compact and has measure 0, then A has content 0.*

Proof. This easily follows from the fact that any open cover of a compact set has a finite subcover, which is just the definition of compactness. ∎

2.2. **A characterisation of integrable functions.** In this subsection, we will determine exactly which functions are integrable. We begin with some auxiliary results on *oscillations*.

Definition 2.6. Let A be a metric space, and let $x_0 \in A$ be a limit point of A. Let $f : A \to \mathbb{R}$ be a bounded function. Let $B(x_0, \delta)$ be ball centered at x_0. Define

$$M(f, x_0, \delta) := \sup_{x\in B(x_0,\delta)} f(x)$$
$$m(f, x_0, \delta) := \inf_{x\in B(x_0,\delta)} f(x)$$

The *oscillation* of f at x_0 is defined as

$$o(f, x_0) = \lim_{\delta\to 0} M(f, x_0, \delta) - m(f, x_0, \delta)$$

The oscillation measures to what extent a function is continuous at a given point. Note that the limit by which oscillations are defined always exist (as the function in consideration is bounded).

Proposition 2.5. *Let A, x_0 and f be as above. Then f is continuous at x_0 if and only if $o(f, x_0) = 0$.*

Proof. Clear by the definition of continuity. ∎

Proposition 2.6. *Let $A \subset \mathbb{R}^n$ be closed, and let $f : A \to \mathbb{R}$ be any bounded function. For any $\epsilon > 0$, the set*

$$B_\epsilon = \{x \in A | o(f, x) \geq \epsilon\}$$

is closed.

Proof. Let y be a limit point of B, and since A is closed, $y \in A$ (so that $f(y)$ is well defined). Let $\delta > 0$ be given, and there is some $x \in B_\epsilon$ such that $x \in B(y, \delta) \cap A$. Moreover, there is some δ_1 such that $B(x, \delta_1) \cap A \subset B(y, \delta) \cap A$. Now, because $o(f, x) \geq \epsilon$, it follows that

$$M(f, x, \delta_1) - m(f, x, \delta_1) \geq \epsilon$$

and hence this shows that

$$M(f, y, \delta) - m(f, y, \delta) \geq \epsilon$$

Taking limits as $\delta \to 0$, we get that $o(f, y) \geq \epsilon$, and hence $y \in B_\epsilon$. The proof is complete. ∎

Lemma 2.7. *Let $R \subset \mathbb{R}^n$ be a closed rectangle, and let $f : R \to \mathbb{R}$ be a bounded function such that $o(f, x) \leq \epsilon$ for every $x \in R$. Then, there is some partition P of R such that*

$$U(P, f) - L(P, f) < \epsilon v(R)$$

Proof. To be completed. ■

Theorem 2.8. *Let $R \subset \mathbb{R}^n$ be a closed rectangle, and let $f : R \to \mathbb{R}$ be a bounded function. Let*

$$B = \{x \in R | f \text{ is not continuous at } x\}$$

Then f is integrable on R if and only if B has measure 0. In other words, f is integrable if and only if its set of discontinuities has measure 0.

Proof. To be completed. ■

2.3. **Extension of integrability to subsets of Rectangles.** Having studied integration on rectangles, we can now define the notion of integration to subsets of rectangles. We do this using *characteristic functions* of sets.

Definition 2.7. Let R be a closed rectangle, and let $f : R \to \mathbb{R}$ be a bounded function. Let $C \subset R$. Define

$$\int_C f = \int_R f \cdot \chi_C$$

given that $f \cdot \chi_C$ is integrable on R.

So, integrals on a subset C will certainly be defined if both f and χ_C are integrable on R (as product of integrable functions is integrable). We now find a criteria to see when χ_C is integrable. Before that, we verify a quick property.

Proposition 2.9. *Let C be a set such that there is some closed rectangle $R \subset \mathbb{R}^n$ such that χ_C is integrable on R. Then, for any closed rectangle R' containing C, χ_C is integrable on R'.*

Proof. Suppose C is a set such that R is a closed rectangle containing C such that χ_C is integrable on R. Let R' be any other closed rectangle containing C. Consider $R \cap R'$, which is again a closed rectangle containing C. Let P be a partition of R such that $R \cap R'$ is a subrectangle of the partition. Then we know that χ_C is integrable on this subrectangle, and that

$$\int_R \chi_C = \sum_{S \in P, S \neq R \cap R'} \int_S \chi_C + \int_{R \cap R'} \chi_C = \int_{R \cap R'} \chi_C$$

Finally, take any partition S' of R' that contains $R \cap R'$ as a subrectangle. It follows that χ_C is integrable on R'. ■

Theorem 2.10. *Let $C \subset R \subset \mathbb{R}^n$ where R is a closed rectangle. Then, χ_C is integrable on R if and only if ∂C has measure 0 (and hence content 0).*

Proof. Since C is bounded, $\overline{C}$ is also bounded, and take a closed rectangle $R_0 \subset \mathbb{R}^n$ such that $\overline{C} \subset \text{Int}(R_0)$. Now, we show that the set of points of discontinuities of χ_C in R_0 is exactly ∂C. Any point in the interior of C must be a point where χ_C is continuous, and similarly for any point in $(\mathbb{R}^n - \overline{C}) \cap R_0$. Moreover, any point on ∂C is a point of discontinuity, because any neighborhood intersects with C and $(\mathbb{R}^n - C) \cap R_0$. So, χ_C is integrable on R_0 if and only if ∂C has measure 0 (and hence content 0, since ∂C is compact). By **Proposition** 2.9, it follows that χ_C is integrable on R if and only if ∂C has measure (or content) 0. ■

Definition 2.8. Let C be a bounded subset of $\mathbb{R}^n$ such that ∂C has measure 0. Then C is said to be *Jordan Measurable*. The integral $\int_C \mathbf{1}$ is said to be the n-dimensional *volume* of C.

Now, we will try to extend the notion of integrability even further. Above, we started with a function on some rectangle, and we defined the integral on a subset of the rectangle. We now start with a function on some subset of $\mathbb{R}^n$, which is not necessarily a rectangle, but we will assume it to be compact.

Definition 2.9. Any compact subset of S of $\mathbb{R}^n$ with the property that χ_S is integrable on R for some rectangle R containing S is said to be *acceptable*. (*This is not standard terminology*).

Remark 2.10.1. By **Proposition** 2.9, we know that if S is acceptable, then χ_S is integrable on *any* rectangle containing S. By **Theorem** 2.10, we immediately see that a compact set S is acceptable if and only if ∂S has measure (or content) 0.

Definition 2.10. Let S be an acceptable set, and let R be a closed rectangle containing S. Let $f : S \to \mathbb{R}$ be bounded function. Define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{, if } x \in S \\ 0 & \text{, if } x \notin S \end{cases}$$

and then define

$$\int_S f = \int_R \tilde{f}$$

Remark 2.10.2. This definition is different from **Definition** 2.7, because here we start with a function on a set, and extend it to the rectangle. However, as we shall see, both definitions co-incide for *acceptable* sets.

Theorem 2.11 (**Tietze Extension Theorem**)**.** *Let X be a metric space, and let A be a closed subset of X. Suppose $f : A \to \mathbb{R}$ is a continuous function. Then, f can be extended to X, i.e there is some continuous $g : X \to \mathbb{R}$ such that $g|_A = f$.*

Theorem 2.12. *Let S be any acceptable set. Given any closed rectangle R such that $S \subset R$, the restriction of a Riemann integrable function on R to S is Riemann-integrable. Specifically, if g is integrable on R, then*

$$\int_R g(x)\chi_S(x) = \int_S g|_S$$

Moreover, continuous functions on S are integrable (where S is interpretted as a metric space in itself).

Proof. The first two statements follow by the equation

$$g|_S = g \cdot \chi_S$$

and the fact that the product of integrable functions is integrable. For the second statment, observe that by the **Tietze Extension Theorem** 2.11, any continuous map from S to $\mathbb{R}$ can be extended to one from R to $\mathbb{R}$, and we know that continuous functions are integrable. ∎

3. Iterated Integrals and Fubini's Theorem

First, some intuition. Suppose $f : R = [a,b] \times [c,d] \to \mathbb{R}$ is a continuous function. Then,

$$\int_R f$$

is just the volume under the surface described by f. Now, if $x \in [a,b]$, we get a function $f_x : [c,d] \to \mathbb{R}$ defined by

$$f_x(y) = f(x,y)$$

It is then reasonable to assume that the volume under the surface is

$$\int_a^b \left(\int_c^d f_x(y)dy \right) dx$$

Moreover, if we switched the roles of the variables x, y, it is reasonable that the answer will be the same. This actually turns out to be true, and we will prove a general version of this process, which will lead to the famous theorem given below.

Theorem 3.1 (**Fubini's Theorem**). *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be two closed rectangles, and let*

$$f : A \times B \to \mathbb{R}$$

be an integrable function on the product rectangle $A \times B$ in $\mathbb{R}^{n+m}$. For $x \in A$, define

$$g_x(y) = f(x,y)$$

so that $g_x : B \to \mathbb{R}$. Define

$$\mathcal{L}(x) = \underline{\int_B} g_x$$

and

$$\mathcal{U}(x) = \overline{\int_B} g_x$$

i.e $\mathcal{L}(x)$ and $\mathcal{U}(x)$ are the lower and upper integrals of g_x over B (note that g_x need not be integrable). Then, $\mathcal{L}$ and $\mathcal{U}$ are integrable over A, and

$$\int_A \mathcal{L} = \int_A \mathcal{U} = \int_{A \times B} f$$

Proof. Let P_A be a partition of A, P_B be a partition of B, so that $P_A \times P_B$ is a partition of $A \times B$. Any subrectangle S in $P_A \times P_B$ is of the form

$$S_A \times S_B$$

where S_A is a subrectangle in P_A and S_B is a subrectangle in P_B. First, the inequality

$$\mathcal{L}(x) \leq \mathcal{U}(x)$$

is clear for any $x \in A$.

A typical lower sum for approximating $\int_{A\times B} f$ will look like

$$\begin{aligned}
\sum_{S_A\in P_A, S_B\in P_B} m_{S_A\times S_B}(f)\text{vol}(S_A\times S_B) &= \sum_{S_A\in P_A, S_B\in P_B} m_{S_A\times S_B}(f)\text{vol}(S_A)\text{vol}(S_B)\\
&= \sum_{S_A\in P_A}\text{vol}(S_A)\sum_{S_B\in P_B} m_{S_A\times S_B}(f)\text{vol}(S_B)\\
&\le \sum_{S_A\in P_A}\text{vol}(S_A)\sum_{S_B\in P_B} m_{S_B}(g_x)\text{vol}(S_B)\\
&\le \sum_{S_A\in P_A}\text{vol}(S_A)\mathcal{L}(x)\\
&\le L(\mathcal{L}, P_A)
\end{aligned}$$

where above x was any point in S_A. Similarly, we can get the inequality

$$\sum_{S_A\in P_A, S_B\in P_B} M_{S_A\times S_B}(f)\text{vol}(S_A\times S_B) \ge U(\mathcal{U}, P_A)$$

Combining these inequalities, we see that

$$L(f, P_A\times P_B) \le L(\mathcal{L}, P_A) \le U(\mathcal{L}, P_A) \le U(\mathcal{U}, P_A) \le U(f, P_A\times P_B)$$

and hence we see that

$$\int_A \mathcal{L} = \int_{A\times B} f$$

Now, it is easy to see that

$$L(\mathcal{L}, P_A) \le L(\mathcal{U}, P_A)$$

and hence

$$\int_A \mathcal{L} = \int_A \mathcal{U} = \int_{A\times B} f$$

which completes the proof. ∎

Remark 3.1.1. Note that, in this proof, we didn't use any new ideas, just compared Riemann sums with each other. This is one of those powerful theorems which have natural proofs.

This theorem has a bunch of important consequences, which are given below.

Corollary 3.1.1. *Let $f : A\times B \to \mathbb{R}$ be Riemann Integrable, where $A\subset\mathbb{R}^n$ and $B\subset\mathbb{R}^m$ are rectangles. Then,*

$$\int_{A\times B} f = \int_B \mathcal{L}' = \int_B \mathcal{U}'$$

where $\mathcal{L}'$ and $\mathcal{U}'$ are defined as the lower and upper integrals of $g_y : A\to\mathbb{R}$.

Proof. This is analogous to the proof of **Fubini's Theorem** 3.1, where instead of freezing the variable x, we freeze the variable y. ∎

Corollary 3.1.2. *Suppose f is integrable on $A\times B$, and in addition suppose each function g_x is integrable on A and g_y is integrable on B for all $x, y\in A, B$ respectively. Then,*

$$\int_{A\times B} f = \int_A\int_B f(x,y)dydx = \int_B\int_A f(x,y)dxdy$$

i.e the two iterated integrals are equal. In particular, this theorem applies when f is continuous.

Example 3.1. (This is problem **3-26** of Spivak's book). As an exercise, we will show that the area under the graph of an integrable function is the one-dimensional integral of the function. Note that now we do have a definition of area to work with.

Let $f : [a,b] \to \mathbb{R}$ be continuous and non-negative and let

$$A_f := \{(x,y) \mid x \in [a,b], 0 \leq y \leq f(x)\}$$

We show that A_f is a Jordan Measurable subset of $\mathbb{R}^2$, and

$$\text{area}(A_f) = \int_a^b f(x)dx$$

and this is the usual interpretation of the integral as the *area under the curve*. It is clear that A_f is a closed and bounded set, so that it is compact. Moreover, we immediately see that the boundary of A_f is

$$\partial A_f = \{a\} \times [0, f(a)] \cup \{b\} \times [0, f(b)] \cup [a,b] \times \{0\} \cup G_f$$

where G_f is the *graph* of f. The first three sets in this union are line segments, so that they have measure (and content) zero. It is enough to show that G_f has measure (and content) zero. But this is clear, because f is Riemann Integrable, and we can find a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) < \epsilon$$

where $\epsilon > 0$ is given and hence we can cover G_f by finitely many rectangles of arbitrarily small area. Thus, A_f is a Jordan Measurable set (and infact, by our definition, an acceptable set). Put $M = \sup_{x\in[a,b]} f(x)$. So it follows that the rectangle $[a,b] \times [0,M]$ contains the set A_f. So by Fubini's Theorem we see that

$$\text{area}(A_f) = \int_{[a,b]\times[0,M]} \chi_A = \int_a^b \mathcal{U}(x)dx$$

where $\mathcal{U} : [a,b] \to \mathbb{R}$ is defined as

$$\mathcal{U}(x) = \overline{\int_0^M} \chi_A(x,y)dy = \int_0^{f(x)} 1dy = f(x)$$

and so we see that

$$\text{area}(A_f) = \int_a^b f(x)dx$$

which completes the proof.

4. Differential Calculus

In this section, we will revise some basic notions of differentiation in higher dimensions. In most of the places, I won't be including a proof.

Definition 4.1. Let $f : U \to \mathbb{R}^m$ be a function, where $U \subset \mathbb{R}^n$ is open. Let $p \in U$, and let $u \in \mathbb{R}^n$ be any vector. Then, the *directional derivative* of f at p along u is defined as

$$D_u f(p) = \lim_{t\to 0} \frac{f(p+tu) - f(p)}{t}$$

whenever the above limit exists.

From differential calculus in higher dimensions, we know the following.

Proposition 4.1. *Suppose $f : U \to \mathbb{R}^m$ is as above, and suppose f is differentiable at p. Then, $D_u f(p)$ exists for all $u \in \mathbb{R}^n$, and*

$$D_u f(p) = Df(p)(u)$$

where $Df(p)$ is the derivative of f at p. In particular, if $m = 1$, then

$$D_u f(p) = \langle \nabla f, u \rangle$$

where $\langle ., . \rangle$ is the standard dot product in $\mathbb{R}^n$.

A characterisation of $\mathscr{C}^1$ maps is relatively straightforward.

Theorem 4.2. *Let $f : U \to \mathbb{R}^m$ be a map, where $U \subset \mathbb{R}^n$ is open. Then, f is $\mathscr{C}^1$ on U if and only if each partial derivative $D_i f_j$ exists and is continuous on U. Usually, the notation is*

$$D_i f_j = \frac{\partial f_j}{\partial x_i}$$

4.1. **A special case of the Chain Rule.** In this short section, we will discuss an important case of the chain rule.
Let $f : U \to \mathbb{R}$ be a $\mathscr{C}^1$ map, where $U \subset \mathbb{R}^2$ is an open set. We use the notation x_1, x_2 for variables instead of x, y. Let $\gamma_1, \gamma_2 : V \to \mathbb{R}$ be $\mathscr{C}^1$ maps, where V is also an open set. These maps will be used to parametrise the variables x_1, x_2, as we will see. Observe that the map $\gamma : V \to U$ given by

$$\gamma(t_1, t_2) = (\gamma_1(t_1, t_2), \gamma_2(t_1, t_2))$$

is a $\mathscr{C}^1$ map as well. So, in simpler words, if a point q in V has coordinates (t_1, t_2), then the point $\gamma(q)$ in U will have coordinates $(\gamma_1(t_1, t_2), \gamma_2(t_1, t_2))$. Finally, let $g : V \to \mathbb{R}$ be the composite map

$$g = f \circ \gamma$$

.
Our goal is to compute the partial derivatives of g with respect to those of f. Observe that, by the chain rule,

$$Dg = Df \circ D\gamma$$

at any point in V. In other words, for any point $q \in V$, put $p = \gamma(q)$, we have

$$\begin{bmatrix} \frac{\partial g}{\partial t_1}(q) & \frac{\partial g}{\partial t_2}(q) \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x_1}(p) & \frac{\partial f}{\partial x_2}(p) \end{bmatrix} \begin{bmatrix} \frac{\partial \gamma_1}{\partial t_1}(q) & \frac{\partial \gamma_1}{\partial t_2}(q) \\ \frac{\partial \gamma_2}{\partial t_1}(q) & \frac{\partial \gamma_2}{\partial t_2}(q) \end{bmatrix}$$

and this immediately gives us the following two equations.

$$\frac{\partial g}{\partial t_1}(q) = \frac{\partial f}{\partial x_1}(p)\frac{\partial \gamma_1}{\partial t_1}(q) + \frac{\partial f}{\partial x_2}(p)\frac{\partial \gamma_2}{\partial t_1}(q)$$
$$\frac{\partial g}{\partial t_2}(q) = \frac{\partial f}{\partial x_1}(p)\frac{\partial \gamma_1}{\partial t_2}(q) + \frac{\partial f}{\partial x_2}(p)\frac{\partial \gamma_2}{\partial t_2}(q)$$

Example 4.1. We now consider an example of the above situation. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathscr{C}^1$ function. Let $\gamma : V \to U$ be a *change of coordinates map*, i.e

$$\gamma(q) = Aq^t$$

for $q, \in V$, where

$$A = \begin{bmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{bmatrix}$$

is an invertible matrix. In this case, we have

$$\gamma_1(t_1, t_2) = a_1^1 t_1 + a_1^2 t_2$$
$$\gamma_2(t_1, t_2) = a_2^1 t_1 + a_2^2 t_2$$

From the above equations, we get for any $q \in V$

$$\frac{\partial g}{\partial t_1}(q) = a_1^1 \frac{\partial f}{\partial x_1}(p) + a_1^2 \frac{\partial f}{\partial x_2}(p)$$
$$\frac{\partial g}{\partial t_2}(q) = a_2^1 \frac{\partial f}{\partial x_1}(p) + a_2^2 \frac{\partial f}{\partial x_2}(p)$$

where again $p = \gamma(q)$.

4.2. **Polar Coordinates.** Let $(x, y) \in \mathbb{R}^2$ such that $(x, y) \neq (0, 0)$. We know that

$$(x, y) = (r \cos\theta, r \sin\theta)$$

for some $r > 0$, and some $\theta \in \mathbb{R}$. These coordinates are called the *polar coordinates* of (x, y). Moreover, r is unique, while θ is unique modulo 2π.
So, consider the following open sets.

$$V := \{(r, \theta) \in \mathbb{R}^2 : r > 0\}$$
$$U := \{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$$

Let $\gamma : V \to U$ be the following map.

$$\gamma(r, \theta) = (r \cos\theta, r \sin\theta)$$

Observe that γ is surjective, not injective, and is a $\mathscr{C}^1$ map. Moreover, at any point $q = (r, \theta) \in V$, we have the following.

$$D\gamma(q) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

So we have that $\det(D\gamma(q)) = r > 0$, so that $D\gamma(q)$ is invertible. Applying the inverse function theorem, we see that there is an open neighborhood V_1 of q contained in V, and an open neighborhood U_1 of $\gamma(q)$ contained in U such that $\gamma|_{V_1}$ is injective, and $\gamma(V_1) = U_1$, and that $\gamma^{-1} : U_1 \to V_1$ is also $\mathscr{C}^1$. Moreover, if $q = (r, \theta)$, then we know that

$$D\gamma^{-1}(\gamma(q)) = (D\gamma(q))^{-1}$$

so in matrix form, we have

$$D\gamma^{-1}(r\cos\theta, r\sin\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix}$$

where the inverse is calculated using Cramer's rule.

Example 4.2. As an example of the above procedure, we can compute the *Laplacian* of a map. So, let $f : U \to \mathbb{R}$ be a $\mathscr{C}^2$ map for some open subset $U \subset \mathbb{R}^2$. The *Laplacian* of f at a point is defined as

$$\Delta f = -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}$$

Fix $p \in U$, and let $q = \gamma^{-1}(p)$, where say θ is chosen modulo 2π. As above, the inverse function theorem gives us an open neighborhood V of q such that $\gamma|_V$ is injective and $\gamma(V)$ is an open subset of $\mathbb{R}^2$ containing p. So, without loss of generality, let $U = \gamma(V)$. Observe that on U, we have

$$f = g \circ \gamma^{-1}$$

and the chain rule immediately gives us the following two equations.

$$\frac{\partial f}{\partial x}(p) = \frac{\partial g}{\partial r}(q) \cos\theta - \frac{\partial g}{\partial \theta}(q)\frac{\sin\theta}{r}$$
$$\frac{\partial f}{\partial y}(p) = \frac{\partial g}{\partial r}(q) \sin\theta + \frac{\partial g}{\partial \theta}(q)\frac{\cos\theta}{r}$$

Using these equations, the Δf can be computed in polar coordinates, and one sees that

$$\Delta f(p) = -\frac{\partial^2 g}{\partial r^2}(q) - \frac{1}{r}\frac{\partial g}{\partial r}(q) - \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}(q)$$

5. Partitions of Unity

In this section, we will study a very important tool which is used throughout analysis. Let us begin by proving a theorem.

Theorem 5.1. *Let $A \subset \mathbb{R}^n$ be any set, and let $\mathcal{O}$ be an open cover of A. Then, there is a family Φ of $\mathscr{C}^\infty$ functions φ defined in an open set containing A having the following properties.*

(1) *$\varphi \geq 0$ for each $\varphi \in \Phi$.*
(2) *For each $x \in A$, there is an open neighborhood V of x such that all but finitely many $\varphi \in \Phi$ are 0 on V.*
(3) *For each $x \in A$, the following sum holds:*

$$\sum_{\varphi \in \Phi} \varphi(x) = 1$$

and note that by (2)*, this sum is finite.*

(4) *For each $\varphi \in \Phi$, there is an open set $U \in \mathcal{O}$ such that the support of φ is a compact subset of U.*

Remark 5.1.1. This version of the theorem is also available in Spivak's book, but with an error. In (4), Spivak says that the support of φ is *closed*, but instead it should say *compact*. Similarly, in the third line of page 64, in the line *which is 1 on A and 0 outside of some closed set in U*, the word *closed* must be replaced by *compact*. See this link and related threads for more information.

Definition 5.1. If the family Φ satisfies (1)-(3), then it is called a $\mathscr{C}^\infty$ *partition of unity* for A. If Φ also satisfies (4), then it is said to be *subordinate* to the cover $\mathcal{O}$.

Proof of **Theorem** 5.1. To be completed (For the first time, it is fine to skip this proof, because the proof is not important for our purpose of applying these.) ■

Corollary 5.1.1. *Let A, $\mathcal{O}$, and Φ be as above, and let $C \subset A$ be a compact set. Then, all but finitely many $\varphi \in \Phi$ are zero on C.*

Proof. Let $x \in C$, and hence there is some open set V_x such that all but finitely many $\varphi \in \Phi$ are zero on V_x. Then, the cover

$$\bigcup_{x \in C} V_x$$

is an open cover of C, and hence there is some finite subcover

$$C \subset V_{x_1} \cup ... \cup V_{x_n}$$

It is then clear that all but finitely many $\varphi \in \Phi$ are zero on C. ∎

Corollary 5.1.2. *Let $K \subset \mathbb{R}^n$ be a compact set, and let $\mathcal{O}$ be an open cover of K. Then, there exist finitely many $\mathscr{C}^\infty$ functions $\varphi_1, ..., \varphi_k$ such that $\varphi_i \geq 0$ for each i, each φ_i has compact support in some $U_i \in \mathcal{O}$ and $\sum_i \varphi_i(x) = 1$ for each $x \in K$.*

Definition 5.2. Let $A \subset \mathbb{R}^n$ be an open set, and let $\mathcal{O}$ be an open cover of A. $\mathcal{O}$ is said to be an *admissible cover* of A if each $U \in \mathcal{O}$ is contained in A.

6. Change Of Variables

In this section, we will derive the change of variables formula for integrals in higher dimensions. First, we start with some preliminary concepts (a lot of these ideas are taken from Baby Rudin).

6.1. **Primitive Mappings.**

Informally, a primitive mapping is one which leaves all but one variable fixed. We now describe these more formally.

Definition 6.1. A mapping $G : U \to \mathbb{R}^n$ for some open subset $U \subset \mathbb{R}^n$ is said to be a *primitive mapping* if G can be written as

$$G(x_1, ..., x_m, ..., x_n) = (x_1, ..., g(x_1, ..., x_n), ..., x_n)$$

for some function $g : U \to \mathbb{R}$, i.e we are changing the m^{th} coordinate and keeping other coordinates fixed.

Observe that if g is differentiable at a point $a \in U$, it is clear that G is also differentiable at that point. Moreover, the Jacobian matrix of G at the point a will look something like below.

$$(JG)(a) = \begin{bmatrix} 1 & 0 & 0 & .... & 0 \\ 0 & 1 & 0 & .... & 0 \\ 0 & 0 & 1 & .... & 0 \\ ... & ... & ... & .... & ... \\ D_1g(a) & D_2g(a) & D_3g(a) & ... & D_ng(a) \\ ... & ... & ... & .... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}$$

i.e $(JG)(a)$ is the identity matrix, but the m^{th} row is $\nabla g(a)$, and hence it is clear that

$$\text{det}(JG)(a) = D_m g(a)$$

so that $G'(a)$ is invertible if and only if $D_m g(a) \neq 0$.

Definition 6.2. A linear map B on $\mathbb{R}^n$ is said to be a *flip* if it interchanges two coordinates (and these are also called *transpositions* as permutations).

As we see in the following theorem, any $\mathscr{C}^1$ mapping locally be written as a product of primitive maps and flips.

Theorem 6.1. *Let $F : U \to \mathbb{R}^n$ be a $\mathscr{C}^1$ map for some open set $U \subset \mathbb{R}^n$ and let $t_0 \in U$ with $F'(t_0)$ invertible. There exists a permutation B of the coordinates and (for $i = 1, .., n$) primitive $\mathscr{C}^1$ maps $G_i : U_i \to \mathbb{R}^n$ (where U_i is an open neighborhood of the origin) such that*

$$G_i(0) = 0,\ \ G_i(U_i) \subset U_{i-1}$$

such that

(1) $U_n + t_0 \subset U$ *and*
(2) *if* $t \in U_n$*, then*

$$F(t + t_0) = B \circ G_1 \circ ... \circ G_n(t) + F(t_0)$$

(3) $G_i'(0)$ *is invertible.*

and so in simpler words, we have written F locally around t_0 as a product of a permutation and primitive mappings. Note that B can be written as a product of flips.

Proof. To be completed. For now, maybe it is fine to understand the theorem and skip the proof. This is theorem 10.7 in Baby Rudin. ■

6.2. **The Change of Variables Formula.** In this section, we will state and prove a version of the *change of variables* formula in higher dimensional integration. First, let us look at a version of the change of variables formula in one dimension.

Proposition 6.2. *Let $\gamma : (c, d) \to (a, b)$ be a $\mathscr{C}^1$ bijection with $\mathscr{C}^1$ inverse. Let $I = [a_0, b_0]$ be a closed interval contained in (a, b), and let f be a continuous function on (a, b). Then,*

$$\int_I f(x)dx = \int_{\gamma^{-1}(I)} f(\gamma(t))|\gamma'(t)|dt$$

Remark 6.2.1. We note that this is not the most general situation in which a change of variables can be done in one variable. But the point here is to generalise to higher dimensions, and so we only deal with this special case.

Remark 6.2.2. The absolute value around $\gamma'(t)$ has the explanation that γ can either be *increasing* or *decreasing*, and if we don't put the absolute value sign, the equation won't be true in the case when γ is *decreasing*. So handle both cases, we include the absolute value (Try some examples yourself).

Proof. As mentioned in the above remarks, we will handle two cases, one where γ is increasing and the other where it is decreasing. A point to be noted is that because γ has a $\mathscr{C}^1$ inverse, we get by the chain rule that $\gamma'(t)$ cannot be zero for any $t \in (c, d)$.

- First, suppose γ is increasing, and since it is a bijection, it is strictly increasing. Also, $\gamma'(t) > 0$ for every $t \in (c, d)$, and hence

$$|\gamma'(t)| = \gamma(t)$$

for every $t \in (c,d)$. So suppose $\gamma[c_0,d_0] = [a_0,b_0]$ where $[c_0,d_0] \subset (c,d)$. Define the function $F : [a_0,b_0] \to \mathbb{R}$ by

$$F(x) = \int_{a_0}^{x} f(t)dt$$

and it is clear that $F'(x) = f(x)$ for each $x \in [a_0,b_0]$. By the Fundamental Theorem of Calculus, we know that

$$\int_{a_0}^{b_0} f(t)dt = F(b_0) - F(a_0) = F(\gamma(d_0)) - F(\gamma(c_0))$$

Also, consider the function $F \circ \gamma$ on $[c_0,d_0]$. We know that

$$(F \circ \gamma)'(t) = f(\gamma(t))\gamma'(t) = f(\gamma(t))|\gamma'(t)|$$

for any $t \in [c_0,d_0]$. Again by the Fundamental Theorem of Calculus, we see that

$$\int_{c_0}^{d_0} f(\gamma(t))|\gamma'(t)|dt = F(\gamma(d_0)) - F(\gamma(c_0))$$

and so we have

$$\int_I f(t)dt = \int_{\gamma^{-1}(I)} f(\gamma(t))|\gamma'(t)|dt$$

- In the second case, γ is decreasing. Exactly the same proof works as above, but with sign changes. So this completes the proof of this case as well.

■

Remark 6.2.3. We can remedy the above situation by giving an *orientation* to intervals in $\mathbb{R}$. We will see this viewpoint when we do integration on forms.

Definition 6.3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function with compact support. Let $R \subset \mathbb{R}^n$ be any rectangle containing the support of f. Then, define

$$\int_{\mathbb{R}^n} f = \int_R f$$

and we know from **Proposition** 2.9 that the choice of the rectangle R is immaterial.

Theorem 6.3. *Suppose γ is a $1-1$ $\mathscr{C}^1$ mapping of an open set $V \subset \mathbb{R}^n$ into $\mathbb{R}^n$ such that* $\det(J\gamma)(x) \neq 0$ *for all $x \in V$. Let f be a continuous mapping on $\mathbb{R}^n$ with compact support contained in $\gamma(V)$. Then*

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} f(\gamma(x))|\det(J\gamma)(x)|dx \tag{†}$$

Before proving this theorem, we mention a couple of important observations.

Remark 6.3.1. Since γ is a $1-1$ $\mathscr{C}^1$ mapping with invertibles derivatives at every point, it can be shown that $\gamma : V \to \gamma(V) = U$ is actually a $\mathscr{C}^1$ *diffeomorphism* (the proof is simple and uses the inverse function theorem. Try it). In particular, $\gamma(V) = U$ is an open set, and $\operatorname{supp} f \subseteq U$.

Remark 6.3.2. From the above remark, γ is a diffeomorphism. Define the function $h : \mathbb{R}^n \to \mathbb{R}^n$

$$h(x) = \begin{cases} 0 & , \quad x \in V^c \\ f(\gamma(x))|\det(J\gamma)(x)| & , \quad x \in V \end{cases}$$

We see that h is a continuous function, and in particular we have

$$\text{supp}(h) = \gamma^{-1}(\text{supp}(f)) \subset V$$

because $\gamma : V \to U$ is a diffeomorphism (and we are using the fact that $|\det(J\gamma)(x)| \neq 0$ at any point $x \in V$). It is then clear that the integrand in the RHS of equation (†) has compact support, and hence the integral is well-defined.

We will now try to prove **Theorem** 6.3 in a sequence of steps.

Proposition 6.4 (**Step 1**). *If the statement of* **Theorem** 6.3 *holds for one-one $\mathscr{C}^1$ maps $\gamma : V \to U$ and $\gamma_1 : V_1 \to U$, then it holds for the one-one $\mathscr{C}^1$ map $\gamma \circ \gamma_1 : V_1 \to U$. Here, all the sets V_1, V, U are open subsets of $\mathbb{R}^n$.*

Proof. Let f be a continuous function with compact support such that supp$f \subset U$. Applying **Theorem** 6.3 to the map γ, we see that

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} f(\gamma(x))|\det(J\gamma)(x)|dx$$

where by **Remark** 6.3.2, the function

$$f(\gamma(x))|\det(J\gamma)(x)|$$

on V is a continuous function with compact support contained in V. So, applying **Theorem** 6.3 again with this function and the map γ_1, we see that

$$\begin{aligned}\int_{\mathbb{R}^n} f(\gamma(x))|\det(J\gamma)(x)|dx &= \int_{\mathbb{R}^n} f(\gamma(\gamma_1(x)))|\det(J(\gamma)(\gamma_1(x)))||\det(J\gamma_1)(x)|dx \\ &= \int_{\mathbb{R}^n} f(\gamma \circ \gamma_1(x))|\det(J\gamma \circ \gamma_1)(x)|dx\end{aligned}$$

where we used the chain rule in the last step. This proves the claim. ∎

Proposition 6.5 (**Step 2**). **Theorem** 6.3 *holds when γ is a permutation of coordinates (in particular, $\gamma : V \to U$ is the restriction of a permutation of coordinates to V, which we know is an invertible linear map).*

Proof. First, observe that if γ is a permutation of coordinates, then we can write it as a product of flips (or in group theoretic terms, every permutation is a product of transpositions). So, by the help of **Proposition** 6.4, it is enough to prove the case when γ is a flip. Without loss of generality, suppose

$$\gamma(x_1, x_2, ..., x_n) = (x_2, x_1, ..., x_n)$$

so that $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, and we assume without loss of generality that $V = U = \mathbb{R}^n$. Moreover, $J\gamma(x)$ at any point in $\mathbb{R}^n$ is a permutation matrix, and it's determinant is ± 1, so that

$$|\det J\gamma(x)| = 1$$

Now, let R be any rectangle in $\mathbb{R}^n$ containing the support of f, and let $R = [a_1, b_1] \times ... \times [a_n, b_n]$. We have

$$R' = \gamma^{-1}(R) = [a_2, b_2] \times [a_1, b_1] \times [a_3, b_3] ... \times [a_n, b_n]$$

Clearly, R' contains the support of $f \circ \gamma$. Put

$$R'' = [a_3, b_3] \times ... \times [a_n, b_n]$$

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x_1, x_2, x_3, ..., x_n) dx_1 dx_2 ... dx_n &= \int_R f(x_1, x_2, x_3, ..., x_n) dx_1 ... dx_n \\
&= \int_{[a_1,b_1]\times[a_2,b_2]} \left(\int_{R''} f(x_1, x_2, x_3, ..., x_n) dx_3 ... dx_n \right) dx_1 dx_2 \\
&= \int_{[a_2,b_2]} \left(\int_{[a_1,b_1]} \left(\int_{R''} f(x_1, x_2, x_3, ..., x_n) dx_3 .. dx_n \right) dx_1 \right) dx_2 \\
&= \int_{[a_2,b_2]} \left(\int_{[a_1,b_1]\times R''} f(x_1, x_2, x_3, ..., x_n) dx_1 dx_3 ... dx_n \right) dx_2 \\
&= \int_{[a_2,b_2]} \left(\int_{[a_1,b_1]\times R''} f(\gamma(x_2, x_1, x_3, ..., x_n)) dx_1 dx_3 ... dx_n \right) dx_2 \\
&= \int_{R'} f(\gamma(x_2, x_1, ..., x_n)) dx_2 dx_1 ... dx_n \\
&= \int_{\mathbb{R}^n} f(\gamma(x)) |J\gamma(x)| dx
\end{aligned}$$

and this completes the proof. ■

Proposition 6.6 (**Step 3**). **Theorem** 6.3 *holds if γ is a one-one $\mathscr{C}^1$ primitive mapping.*

Proof. This statement is just the one variable change of variables formula in disguise, as we will see. Suppose γ is a primitive one-one $\mathscr{C}^1$ map, and without loss of generality suppose

$$\gamma(x_1, ..., x_n) = (x_1, ..., x_{n-1}, g(x_1, ..., x_n))$$

where $g : V \to \mathbb{R}$ is a $\mathscr{C}^1$ map. Since γ is one-one on V, it follows that g is also one-one on V. By our hypothesis, the support of f is compact and contained in $U = \gamma(V)$. By **Remark** 6.3.2, we see that the function $(f \circ \gamma)|\det J\gamma|$ has compact support contained in V, and hence there is some rectangle $R \subset V$ which contains the support of this function. Also, by the discussion in the section on primitive maps, we see that

$$|\det J\gamma(x)| = \left| \frac{\partial g}{\partial x_n}(x) \right|$$

To be completed. There is a nice trick to complete this argument ■

Proposition 6.7 (**Step 3.5**). **Theorem** 6.3 *holds if γ is a translation (in particular, $\gamma : V \to U$ is the restriction of a translation, and every translation is one-one and $\mathscr{C}^\infty$).*

Proof. Let γ be a translation, i.e

$$\gamma(x) = x + a$$

for some $a \in \mathbb{R}^n$. Clearly, $\gamma : V \to U$ is a one-one $\mathscr{C}^1$ (infact $\mathscr{C}^\infty$) mapping with invertible derivative at every point of V (the derivative being the identity map at every point of V). The claim is that γ can be written as a product of n primitive mappings. In terms of coordinates, we have

$$\gamma(x_1, ..., x_n) = (x_1 + a_1, ..., x_n + a_n)$$

For $1 \leq i \leq n$, let $\gamma_i : \mathbb{R}^n \to \mathbb{R}^n$ be the map

$$\gamma_i(x_1, ..., x_n) = (x_1, ..., x_i + a_i, ..., x_n)$$

and it is clear that γ_i is a one-one $\mathscr{C}^1$ primitive map for each $1 \leq i \leq n$ with invertible derivative. Also, we have

$$\gamma = \gamma_1 \circ \gamma_2 \circ ... \circ \gamma_n$$

(and infact, the order of the product above does not matter). By **Step 3** 6.6 and **Step 1** 6.4, the claim follows. ∎

Proposition 6.8 (**Step 4**)**.** *Let f, γ, V and U be as in the statement of* **Theorem** 6.3*. Let $\{U_\alpha\}$ be an open cover of U where $U_\alpha \subseteq U$ for each α such that the statement of the theorem holds for any continuous function f_α with compact support contained in U_α. Then, the statement of the theorem holds for f as well.*

Proof. We know that supp(f) is compact and is contained in U, and hence $\{U_\alpha\}$ is an open cover of supp(f). By **Corollary** 5.1.2. we know that there are continuous (infact $\mathscr{C}^\infty$) functions $\varphi_1, ..., \varphi_k$ with compact support such that supp$(\varphi_i) \subseteq U_\alpha$ for some α for each $1 \leq i \leq k$ and that

$$\sum_{i=1}^{k} \varphi_i(x) = 1$$

for each $x \in$ supp(f). Moreover, it is easy to see that supp$(\varphi_i f)$ is compact for each $1 \leq i \leq k$ and is contained in U_α for some α (and obviously $\varphi_i f$ is continuous for each $1 \leq i \leq k$). Finally, observe that

$$f = \sum_{i=1}^{k} \varphi_i f$$

Also, observe that if we define

$$\varphi_i' = \varphi_i \circ \gamma$$

then $\varphi_1', ...\varphi_k'$ is a partition of unity for supp$(f \circ \gamma)$ (which is a compact set, see **Remark** 6.3.2) subordinate to the cover $\{V_\alpha\}$ where $V_\alpha = \gamma^{-1}(U_\alpha)$. By the hypothesis of our proposition, the theorem holds true for each $\varphi_i f$. So, we have the following.

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)dx &= \int_{\mathbb{R}^n} \sum_{i=1}^{k} \varphi_i(x)f(x)dx \\
&= \sum_{i=1}^{k} \int_{\mathbb{R}^n} \varphi_i(x)f(x)dx \\
&= \sum_{i=1}^{n} \int_{\mathbb{R}^n} \varphi_i(\gamma(x))f(\gamma(x))|\operatorname{det}(J\gamma)(x)|dx \\
&= \sum_{i=1}^{n} \int_{\mathbb{R}^n} \varphi_i'(x)f(\gamma(x))|\operatorname{det}(J\gamma)(x)|dx \\
&= \int_{\mathbb{R}^n} \sum_{i=1}^{n} \varphi_i'(x)f(\gamma(x))|\operatorname{det}(J\gamma)(x)|dx \\
&= \int_{\mathbb{R}^n} f(\gamma(x))|\operatorname{det}(J\gamma)(x)|dx
\end{aligned}$$

which completes the proof. ■

Final Step for **Theorem** 6.3. Let f, γ, U, V be as in the statement of the theorem. Let $a \in V$ be any point. Then by **Theorem** 6.1 there is an open neighborhood $V_a \subset V$ of a such that

$$\gamma(x) = \gamma(a) + BG_1 \circ G_2 \circ ... \circ G_n(x - a)$$

for all $x \in V_a$, where B is a permutation and each G_i for $1 \leq i \leq n$ is a $\mathscr{C}^1$ mapping satisfying the conditions in **Theorem** 6.1. To be completed. We just need to put everything together. ■

6.3. **Volumes under Linear Isomorphisms.** In this section, we will prove an important result about the action of linear isomorphisms to volumes of sets.

Proposition 6.9. *Let $g_1, g_2, g_3 : \mathbb{R}^n \to \mathbb{R}^n$ be linear maps defined as follows.*

$$\begin{aligned}
g_1(x_1, ..., x_j, ..., x_n) &= (x_1, ..., ax_j, ..., x_n) \\
g_2(x_1, ..., x_k, ..., x_n) &= (x_1, ..., x_k + x_j, ..., x_n) \\
g_3(x_1, ..., x_i, ..., x_j, ..., x_n) &= (x_1, ..., x_j, ..., x_i, ..., x_n)
\end{aligned}$$

where $a \in \mathbb{R}$ in the formula for g_1, $1 \leq j < k \leq n$ in the formula for g_2 (and $x_k + x_j$ is the k^{th} coordinate of $g_2(x_1, ..., x_n)$) and $1 \leq i < j \leq n$ in the formula for g_3. Let U be any rectangle in $\mathbb{R}^n$. Then,

$$v(g_i(U)) = |\operatorname{det} g_i| v(U)$$

where $v(U)$ is the n-dimensional volume of U.

Remark 6.9.1. This is part **(a)** of problem **3-35** in Spivak's book. I have written the maps g_i in coordinate form.

Proof. It is immediately seen that g_3 is a flip (or a transposition), and g_1 is invertible if $a \neq 0$ (these remarks are not important for the proof). Now, it is also easily seen that $g_1(U)$ and $g_3(U)$ are rectangles in $\mathbb{R}^n$ and $|\operatorname{det} g_1| = |a|$ and $|\operatorname{det} g_3| = 1$, and it can be easily seen that the formula is indeed true. So, we only deal with the case for g_2.

Suppose $U = [a_1, b_1] \times ... \times [a_n, b_n]$. Then, one can verify that

$$g_2(U) = \bigcup_{x \in [a_j, b_j]} [a_1, b_1] \times ...\{x\} \times ... \times [a_k + x, b_k + x] \times ... \times [a_n, b_n]$$

and it is not hard to see that this set is Jordan Measurable, as its boundary is a union of line segments in $\mathbb{R}^n$ (to be more geometric, the image is actually a parallelogram in $\mathbb{R}^n$. To understand it better, try the case $n = 2$). Applying **Fubini's Theorem** 3.1, we get

$$v(g_2(U)) = \int_{[a_j, b_j]} \left(\int_{[a_1,b_1] \times ... \times [a_k+x, b_k+x] \times ... \times [a_n, b_n]} 1 \right) dx = v(U)$$

and since $\det(g_3) = 1$, the claim follows (Some details are missing from this proof, but they are relatively easy to fill in. In particular, the application of Fubini's Theorem above is not immediate, but not hard to prove). ■

Proposition 6.10. *Let $S \subset \mathbb{R}^n$ be any acceptable set (i.e a compact Jordan measurable set. Look at* **Definition** *2.9). Let g_i be the invertible mappings as in* **Proposition** *6.9 for $1 \leq i \leq 3$ (i.e we only consider the case when g_1 is invertible, so $a \neq 0$). Then*

$$v(g_i(S)) = |\det g_i| v(S)$$

for $1 \leq i \leq 3$.

Proof. Let R be any rectangle in $\mathbb{R}^n$ containing S, and let P be any partition of R. Let χ_S be the characteristic function of S as usual (which we know is integrable over R since S is acceptable). Then, we know that

$$L(P, \chi_S) = \sum_{R' \in P, R' \subset S} v(R') \leq v(S) \leq \sum_{R' \in P, R' \cap S \neq \phi} v(R') = U(P, \chi_S)$$

and here we have just invoked the definition of the upper and lower sums. Because S is acceptable, we know that

$$(\star) \qquad \sup_P \sum_{R' \in P, R' \subset S} v(R') = \inf_P \sum_{R' \in P, R' \cap S \neq \phi} v(R') = v(S)$$

where the supremum and infimum is taken over all partitions P of R. Moreover, observe that

$$\sum_{R' \in P, R' \subset S} v(g_i(R')) \leq v(g_i(S)) \leq \sum_{R' \in P, R' \cap S \neq \phi} v(g_i(R'))$$

(need to justify this step above. Ultimately this is just saying out loud that the total volume of subsets of a set which intersect only along their boundaries is less than the volume of the set itself) for any partition P of R. By **Proposition** 6.9, it follows that

$$|\det g_i| \sum_{R' \in P, R' \subset S} v(R') \leq v(g_i(S)) \leq |\det g_i| \sum_{R' \in P, R' \cap S \neq \phi} v(R')$$

and from $(\star)$ it follows that

$$v(g_i(S)) = |\det g_i| v(S)$$

and this completes the proof. ■

Proposition 6.11. *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be any invertible linear mapping (i.e an isomorphism), and let $S \subset \mathbb{R}^n$ be an acceptable set. Then $g(S)$ is also an acceptable set.*

Proof. It is clear that $g(S)$ is compact, and hence it is enough to show that $\partial g(S)$ has measure zero. Infact a stronger statement is true, i.e

$$\partial g(S) = g(\partial S)$$

but this is just a property of homeomorphisms; we know that g is a homeomorphism (infact a $\mathscr{C}^\infty$ diffeomorphism). So, the claim is true (again missing a lot of details, but again they are not difficult to fill in). ∎

Theorem 6.12. *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map, and let S be any acceptable set in $\mathbb{R}^n$. Then*

$$v(g(S)) = |\operatorname{det}(g)| v(S)$$

Proof. We know that every invertible map can be written as a product of maps given in **Proposition** 6.9 (need to prove this as a separate proposition! This is just reducing an invertible matrix to row echelon form). So suppose

$$g = g_1 \circ g_2 \circ ... \circ g_k$$

where each g_i is a map of one of the forms given in **Proposition** 6.9. Since S is an acceptable set, we know from **Proposition** 6.10 that

$$v(g_k(S)) = |\operatorname{det} g_k| v(S)$$

Also, by **Proposition** 6.11 we see that $g_k(S)$ is an acceptable set in $\mathbb{R}^n$. Repeating the same procedure now with the set $g_k(S)$ and continuing all the way till g_1, we get that

$$v(g(S)) = v(g_1 \circ ... \circ g_k(S)) = |\operatorname{det} g_1| ... |\operatorname{det} g_k| v(S) = |\operatorname{det} g| v(S)$$

and this completes the proof. ∎

7. Integration On Open Sets

In this section we will see one way of defining the notion of integrable functions on *open sets*. The most elementary open set in $\mathbb{R}^n$ is $\mathbb{R}^n$ itself.

7.1. Integration over all of $\mathbb{R}^n$.

Let us begin with a simple definition.

Definition 7.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative function. f is said to be *integrable* on $\mathbb{R}^n$ if f is integrable on each rectangle $R \subset \mathbb{R}^n$ and if

$$\sup_{R \subset \mathbb{R}^n, R \text{ a rectangle}} \int_R f := \int_{\mathbb{R}^n} f < \infty$$

An analogous definition works when f is everywhere non-positive. A simple fact about this definition can be proven.

Proposition 7.1. *If a non-negative function $f : \mathbb{R}^n \to \mathbb{R}$ is integrable on $\mathbb{R}^n$ then there is a nested sequence $S_1 \subset S_2 \subset ...$ of acceptable sets with $\bigcup_{i=1}^\infty S_i = \mathbb{R}^n$ such that f is integrable on each S_i and*

$$\lim_{i \to \infty} \int_{S_i} f = \int_{\mathbb{R}^n} f$$

Proof. First, suppose f is integrable on $\mathbb{R}^n$. Then, by our definition, it is integrable on every rectangle $R \subset \mathbb{R}^n$. Moreover,

$$\int_{\mathbb{R}^n} f = \sup_{R \subset \mathbb{R}^n} \int_R f < \infty$$

So, there is a sequence $\{R_i\}_{i\in\mathbb{N}}$ of rectangles in $\mathbb{R}^n$ such that

$$\lim_{i\to\infty} \int_{R_i} f = \int_{\mathbb{R}^n} f$$

Put $R'_1 = R_1$. Then inductively, let R'_i be any rectangle containing the rectangles $R'_1, ..., R'_{i-1}$ and R_i. Also, choose the rectangles R'_i so that

$$\bigcup_{i\in\mathbb{N}} R'_i = \mathbb{R}^n$$

and this can be easily done by choosing arbitrarily large rectangles. So by our definition it is clear that

$$R'_1 \subset R'_2 \subset ...$$

and because f is non-negative, we see that

$$\int_{R'_i} f \geq \int_{R_i} f$$

for each $i \in \mathbb{N}$, and hence

$$\lim_{i\to\infty} \int_{R'_i} f = \int_{\mathbb{R}^n} f$$

Since a rectangle is an acceptable set, we see that $S_i = R'_i$ is the required sequence of acceptable sets. ■

Definition 7.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be *any* function (i.e not necessarily non-negative or non-positive). f is said to be *absolutely integrable* if f is integrable on rectangles and $|f|$ is integrable on $\mathbb{R}^n$. Define functions f_+ and f_- by $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$. Then

$$f = f_+ - f_-$$

Proposition 7.2. *Let f be an integrable function on a rectangle $R \subset \mathbb{R}^n$. Then f_+ and f_- as defined above are integrable, and*

$$\int_R f = \int_R f_+ - \int_R f_-$$

Proof. We know that if f is integrable, then $|f|$ is also integrable. The integrability of f_+ and f_- immediately follow from the formulae

$$f_+ = \frac{1}{2}(|f| + f)$$
$$f_- = \frac{1}{2}(|f| - f)$$

and the rest of the claim is clear. ■

Definition 7.3. As a result of **Proposition** 7.2, the integral of any absolutely integrable function f on $\mathbb{R}^n$ is define as

$$\int_{\mathbb{R}^n} f := \int_{\mathbb{R}^n} f_+ - \int_{\mathbb{R}^n} f_-$$

Because f is assumed to be absolutely integrable, both the integrals on the RHS are finite and hence they exist.

7.2. **Integration over arbitrary open subsets of $\mathbb{R}^n$.** In this section, we will use a refined version of partitions of unity to define integrability on general open sets in $\mathbb{R}^n$. First we prove a simple fact.

Proposition 7.3. *Let U be any open set in $\mathbb{R}^n$. Then, there is a countable collection $\{R_i\}_{i\in\mathbb{N}}$ of rectangles contained in U such that the interiors* $\text{Int}(R_i)$ *cover U.*

Proof. Let $x \in U$. Then, there is some $\delta_x > 0$ such that $B(x, \delta_x) \subset U$. Pick a point p_x in $B(x, \delta_x)$ having only rational coordinates such that the following can be done: consider an *open* square $B(p_x, \Delta_x)$ (i.e the square has center p_x and edge length Δ_x) such that p_x has only rational coordinates, Δ_x is a rational number, $x \in B(p_x, \Delta_x)$ and $B(p_x, \Delta_x) \subset B(x, \delta_x) \subset U$ (the fact that this can be done is easy! Very similar to Lindelof's Theorem in $\mathbb{R}$). So,

$$U = \bigcup_{x\in U} B(p_x, \Delta_x)$$

and clearly the above union is countable, because rational numbers are being considered. This proves our claim. ■

We now state a stronger version of **Theorem** 5.1 on partitions of unity without a proof.

Theorem 7.4. *Let U be any open subset of $\mathbb{R}^n$, and let $\{R_i\}_{i\in\mathbb{N}}$ be a collection of rectangles as in* **Proposition** 7.3*. Then, there exists a countable family $\{\varphi_i\}_{i\in\mathbb{N}}$ of continuous functions with compact support such that the following hold.*

1. $0 \leq \varphi_i \leq 1$ *for each $i \in \mathbb{N}$.*
2. *The support of each φ_i is contained in* $\text{Int}(R_i)$ (this is where this version is stronger).
3. *For each $x \in U$, there is a neighborhood U_x of x such that only finitely many φ_i are non-zero on U_x.*
4. *For every $x \in U$ we have*

$$\sum_{i\in\mathbb{N}} \varphi_i(x) = 1$$

Definition 7.4. Let $f : U \to \mathbb{R}$ be a function where U is an open subset of $\mathbb{R}^n$. Let $\{R_i\}_{i\in\mathbb{N}}$ be a collection of rectangles as in **Proposition** 7.3, and let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a partition of unity as in **Theorem** 7.4. f is said to be *absolutely integrable* over U if $f|_{R_i}$ is integrable for each $i \in \mathbb{N}$ and

$$\sum_{i\in\mathbb{N}} \int_{\mathbb{R}_i} \varphi_i(x)|f(x)|dx < \infty$$

and in this case we define

$$\int_U f(x)dx := \sum_{i\in\mathbb{N}} \int_{R_i} \varphi_i(x)f(x)dx$$

The above sum exists as the series is absolutely convergent.

It can be shown that (we won't do this here) that the above definition is independent of the choice of rectangles R_i and the partition of unity φ_i.

Proposition 7.5. *Suppose $U = \mathbb{R}^n$. Then,* **Definition** 7.4 *and* **Definition** 7.3 *coincide.*

Proof. To be completed. ∎

7.3. **Generalised Change of Variable.** We can finally state without proof the general version of the change of variables theorem.

Theorem 7.6 (General Change of Variables). *Let V be any open subset of $\mathbb{R}^n$, and let $\gamma : V \to U$ be a $\mathscr{C}^1$ map with invertible derivative at every point of V (equivalently, γ is a diffeomorphism; see* **Remark** 6.3.1*). If f is any absolutely integrable function on $U = \gamma(V)$, then the function*

$$t \mapsto f(\gamma(t))|(J\gamma)(t)|$$

is absolutely integrable on V and

$$\int_U f(x)dx = \int_V f(\gamma(t))|(J\gamma)(t)|dt$$

8. Integration on Chains

8.1. **Algebraic Preliminaries.** First, we will introduce some common ideas of *multilinear algebra* in vector spaces.

Definition 8.1. Let V be a vector space over $\mathbb{R}$. A multilinear function $T : V^k \to \mathbb{R}$ is called a k-*tensor* on V and the set of all k-tensors is denoted by $\mathscr{T}^k(V)$. On the set $\mathscr{T}^k(V)$ define addition addition and scalar multiplication as

$$(S+T)(v_1, ..., v_k) := S(v_1, ..., v_k) + T(v_1, ..., v_k)$$
$$(cS)(v_1, ..., v_k) := cS(v_1, ..., v_k)$$

for any $S, T \in \mathscr{T}(V)$ and $c \in \mathbb{R}$. These definitions make $\mathscr{T}(V)$ into an $\mathbb{R}$-vector space.

Definition 8.2. Let $S \in \mathscr{T}^k(V)$ and $T \in \mathscr{T}^l(V)$, where V is a vector space over $\mathbb{R}$. Define the *tensor product* $S \otimes T \in \mathscr{T}^{k+l}(V)$ by

$$S \otimes T(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) := S(v_1, ..., v_k) \cdot T(v_{k+1}, ..., v_{k+l})$$

Remark 8.0.1. This definition is *not* the usual definition of tensor products, which is generally defined on modules. Moreover, observe that by our definition, the tensor product is not necessarily commutative.

Proposition 8.1. *Let V be an $\mathbb{R}$-vector space. Then the following hold.*

(1) $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$.
(2) $S \otimes (T_1 + T_2) = S \otimes T_1 + S \times T_2$.
(3) $(aS) \otimes T = S \otimes (aT) = a(S \otimes T)$.
(4) $S \otimes (T \otimes U) = (S \otimes T) \otimes U$.

Proof. For (1),(2) and (3), let $v = (v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) \in V^{k+1}$ be fixed.

(1) We see that

$$(S_1 + S_2) \otimes T(v) = (S_1 + S_2)(v_1, ..., v_k) \cdot T(v_{k+1}, ..., v_{k+l}) = S_1 \otimes T(v) + S_2 \otimes T(v)$$

and this proves (1).

(2) This has a similar proof as in (1).
(3) This has a similar proof as in (1).

Associativity of the tensor product, i.e (4) is also easy to prove. ■

Remark 8.1.1. In a similar fashion, we can define the n-fold tensor product $T_1 \otimes ... \otimes T_n$.

Observe that $\mathscr{T}^1(V)$ is just the dual space V^*. The following theorem relates the dual space to the vector space $\mathscr{T}^k(V)$.

Theorem 8.2. *Let $v_1, ..., v_n$ be a basis for V, and let $\varphi_1, ..., \varphi_n$ be the dual basis, i.e $\varphi_i(v_j) = \delta_{ij}$. Then, the set of all k-fold tensor products*

$$\varphi_{i_1} \otimes ... \otimes \varphi_{i_k} \quad , \quad 1 \leq i_1, ..., i_k \leq n$$

is a basis for $\mathscr{T}^k(V)$, and hence $\dim \mathscr{T}^k(V) = n^k$.

Proof. First, let us show that these k-fold tensors are linearly independent.

$$\sum_{(i_1,...,i_k) \in \{1,...,n\}^k} c_{(i_1,...,i_k)} \varphi_{i_1} \otimes ... \otimes \varphi_{i_k} = 0$$

where $c_{(i_1,...,i_k)} \in \mathbb{R}$. Fix any element $(i_1, ..., i_k) \in \{1, ..., n\}^k$, and consider the element $(v_{i_1}, ..., v_{i_k}) \in V^k$. Then, we see that

$$\left(\sum_{(j_1,...,j_k) \in \{1,...,n\}^k} c_{j_1,...,j_k} \varphi_{j_1} \otimes ... \otimes \varphi_{j_k} \right) (v_{i_1}, ..., v_{i_k}) = c_{(i_1,...,i_k)} = 0$$

and hence this implies linear independence of the given elements. Now, we show that the given elements span $\mathscr{T}^k(V)$. Let $T \in \mathscr{T}^k(V)$. Suppose $(w_1, ..., w_k) \in V^k$ such that

$$w_i = \sum_{j=1}^{n} a_{ij} v_j$$

for each $1 \leq i \leq n$. Then, we see that

$$\begin{aligned} T(w_1, ..., w_k) &= \sum_{(i_1,...,i_k) \in \{1,...,n\}^k} a_{1i_1} ... a_{ki_k} T(v_{i_1}, ..., v_{i_k}) \\ &= \sum_{(i_1,...,i_k) \in \{1,...,n\}^k} T(v_{i_1}, ..., v_{i_k}) \varphi_{i_1} \otimes ... \otimes \varphi_{i_k}(w_1, ..., w_k) \end{aligned}$$

and hence we see that

$$T = \sum_{(i_1,...,i_k) \in \{1,...,n\}^k} T(v_{i_1}, ..., v_{i_k}) \varphi_{i_1} \otimes ... \otimes \varphi_{i_k}$$

implying that the given set spans $\mathscr{T}^k(V)$. This completes the proof. ■

Corollary 8.2.1. *If V is any vector space over $\mathbb{R}$, then $V^k \cong \mathscr{T}^k(V)$.*

Proof. This is immediate by comparing dimensions. But there is an elementary proof for this too. ■

Definition 8.3. Let V, W be vector spaces over $\mathbb{R}$, and let $f : V \to W$ be a linear map. Define the *transpose* $f^* : \mathscr{T}^k(W) \to \mathscr{T}^k(V)$ as

$$f^* T(v_1, ..., v_k) = T(f(v_1), ..., f(v_k))$$

where $T \in \mathscr{T}^k(W)$ and $(v_1, ..., v_k) \in V^k$.

Proposition 8.3. *The transpose as defined above is a linear map. Moreover, for suitable multilinear transformations S, T, we have*

$$f^*(S \otimes T) = f^*S \otimes f^*T$$

Proof. First, let us show that $f^* : \mathscr{T}^k(W) \to \mathscr{T}^k(V)$ is a linear map. Suppose $T_1, T_2 \in \mathscr{T}^k(W)$. Then for any $(v_1, ..., v_k) \in V^k$, we have

$$\begin{aligned} f^*(T_1 + T_2)(v_1, ..., v_k) &= (T_1 + T_2)(f(v_1), ..., f(v_k)) \\ &= T_1(f(v_1), ..., f(v_k)) + T_2(f(v_1), ..., f(v_k)) \\ &= f^*T_1(v_1, ..., v_k) + f^*T_2(v_1, ..., v)k) \end{aligned}$$

and hence this implies that

$$f^*(T_1 + T_2) = f^*T_1 + f^*T_2$$

Now if $c \in \mathbb{R}$ and $T \in \mathscr{T}^k(W)$, then for any $(v_1, ..., v_k) \in V^k$ we have

$$f^*(cT)(v_1, ..., v_k) = (cT)(f(v_1), ..., f(v_k)) = cT(f(v_1), ..., f(v_k)) = cf^*T(v_1, ..., v_k)$$

which implies that

$$f^*(cT) = cf^*T$$

and hence f^* is a linear map.

Now suppose $S \in \mathscr{T}^k(W)$ and $T \in \mathscr{T}^l(W)$, so that $S \otimes T \in \mathscr{T}^{k+l}(W)$. For any $(v_1, ..., v_k, v_{k+1}, ..., v_{k+1}) \in V^{k+l}$, we have

$$\begin{aligned} f^*(S \otimes T)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) &= (S \otimes T)(f(v_1), ..., f(v_k), f(v_{k+1}), ..., f(v_{k+l})) \\ &= S(f(v_1), ..., f(v_k)) \cdot T(f(v_{k+1}), ..., f(v_{k+l})) \\ &= f^*S(v_1, ..., v_k) \cdot f^*T(v_{k+1}, ..., v_{k+l}) \\ &= (f^*S \otimes f^*T)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) \end{aligned}$$

and this implies that

$$f^*(S \otimes T) = f^*S \otimes f^*T$$

and this completes the proof. ∎

Definition 8.4. Let V be a vector space over $\mathbb{R}$. An *inner product* on V is a 2-tensor T such that T is *symmetric*, i.e $T(v, w) = T(w, v)$ for all $v, w \in V$. In addition, such an inner product T is said to be *positive definite* if $T(v, v) > 0$ for all $v \neq 0$. The *usual inner product* on $\mathbb{R}^n$ will be denoted by $\langle , \rangle$.

Remark 8.3.1. This is just the usual definition of an inner product on a real vector space. However, the terminology of tensors makes it much more simpler.

Theorem 8.4. *If T is a positive definite inner product on V, there is a basis $v_1, ..., v_n$ for V such that $T(v_i, v_j) = \delta_{ij}$. Consequently, there is an isomorphism $f : \mathbb{R}^n \to V$ such that $T(f(x), f(y)) = \langle x, y \rangle$ for $x, y \in R^n$. In other words, $f^*T = \langle , \rangle$.*

Proof. The first claim is just the existence of an *orthonormal basis* which is just Gram-Schmidt orthogonalisation, which I will not write the proof for.

Now suppose we are given an orthonormal basis $\{v_1, ..., v_n\}$. The required isomorphism $f : \mathbb{R}^n \to V$ is given by $e_i \mapsto v_i$. This completes the proof. ∎

8.2. **Alternating Forms.** We will now try to generalise the *determinant*, which we know is a multilinear alternating function.

Definition 8.5. A k-tensor $\omega \in \mathscr{T}^k(V)$ is called *alternating* if

$$\omega(v_1, ..., v_i, ..., v_j, ..., v_k) = -\omega(v_1, ..., v_j, ..., v_i, ..., v_k)$$

for all $(v_1, ..., v_k) \in V^k$. In simpler terms, interchanging two arguments flips the sign of the function. The set of all alternating k-tensors is a subspace of $\mathscr{T}^k(V)$, denoted by $\Lambda^k(V)$.

Definition 8.6. If $T \in \mathscr{T}^k(V)$, define $\text{Alt}(T) \in \mathscr{T}^k(V)$ by

$$\text{Alt}(T)(v_1, ..., v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) T(v_{\sigma(1)}, ..., v_{\sigma(k)})$$

where S_k is the permutation group on k elements.

Theorem 8.5. *The operator* Alt *satisfies the following.*

(1) *If $T \in \mathscr{T}^k(V)$, then* $\text{Alt}(T) \in \Lambda^k(V)$, *i.e* $\text{Alt} : \mathscr{T}^k(V) \to \Lambda^k(v)$.
(2) *If $\omega \in \Lambda^k(V)$, then* $\text{Alt}(\omega) = \omega$.
(3) *If $T \in \mathscr{T}^k(V)$, then* $\text{Alt}(\text{Alt}(T)) = \text{Alt}(T)$.

Proof. Let us first prove (1). So, let $T \in \mathscr{T}^k(V)$. That $\text{Alt}(T)$ is a k-tensor is clear, and so we only need to show that it is an *alternating* k-tensor. So let $1 \leq i < j \leq n$. For any $\sigma \in S_k$, consider the linear isomorphism $\sigma : V^k \to V^k$ given by

$$\sigma(v_1, ..., v_k) = (v_{\sigma(1)}, ..., v_{\sigma(k)})$$

Now, let τ be the linear isomorphism that flips the i^{th} and j^{th} coordinates. So we have the following chain of equations.

$$\begin{aligned}
\text{Alt}(T)(v_1, ..., v_j, ..., v_i, ..., v_n) &= \text{Alt}(T)(\tau(v_1, ..., v_i, ..., v_j, ..., v_n)) \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) T(\sigma\tau(v_1, ..., v_i, ..., v_j, ..., v_n)) \\
&= \frac{-1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma\tau) T(\sigma\tau(v_1, ..., v_i, ..., v_j, ..., v_n))
\end{aligned}$$

Now we know that the map $S_k \to S_k$ given by $\sigma \mapsto \sigma\tau$ is a bijection. So, the last sum is equal to $-\text{Alt}(T)(v_1, ..., v_i, ..., v_j, ..., v_n)$. This shows that $\text{Alt}(T) \in \Lambda^k(V)$, completing the proof of (1).

Now we prove (2). So suppose $\omega \in \Lambda^k(V)$. So for any $\sigma \in S_k$, we see that

$$\omega(\sigma(v_1, ..., v_n)) = \text{sgn}(\sigma)\omega(v_1, ..., v_n)$$

because σ can be written as a product of transpositions. Hence, we have

$$\begin{aligned}
\text{Alt}(\omega)(v_1, ..., v_n) &= \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)\omega(\sigma(v_1, ..., v_n)) \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \omega(v_1, ..., v_n) \\
&= \omega(v_1, ..., v_n)
\end{aligned}$$

and this proves (2). (3) follows immediately from (1) and (2), and this completes the proof. ∎

Definition 8.7. Let ω, η be any two tensors in $\mathscr{T}^k(V)$. Define the *wedge product* $w \wedge \eta$ by

$$w \wedge \eta = \frac{(k+l)!}{k!l!}\text{Alt}(\omega \otimes \eta)$$

It is clear that $\omega \wedge \eta \in \Lambda^k(V)$.

Proposition 8.6. *The wedge product satisfies the following properties.*

(1) $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$.
(2) $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$.
(3) $a\omega \wedge \eta = \omega \wedge a\eta = a(\omega \wedge \eta)$.
(4) $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.
(5) $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$.

Proof. The first three properties are immediate from the basic fact that Alt is a linear map. So I won't prove those.

So we prove (4) first. Let k, l be positive integers and let $\omega \in \mathscr{T}^k(V), \eta \in \mathscr{T}^l(V)$. Now let $\tau \in S_{k+l}$ be the permutation given by

$$\{1, ..., k, k+1, ..., k+l\} \mapsto \{k+1, ..., k+l, 1, ..., k\}$$

It is not hard to see that τ can be written as a product of kl transpositions, and hence

$$\text{sgn}(\tau) = (-1)^{kl}$$

We now have the following chain of equations.

$$\begin{aligned}
\frac{(k+l)!}{k!l!}\text{Alt}(\omega \otimes \eta)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) &= \frac{(k+l)!}{k!l!}\frac{1}{(k+l)!}\sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma)(\omega \otimes \eta)(\sigma(v_1, ..., v_{k+l})) \\
&= \frac{1}{k!l!}\sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma)\omega(v_{\sigma(1)}, ..., v_{\sigma(k)})\eta(v_{\sigma(k+1)}, ..., v_{\sigma(k+l)}) \\
&= \frac{1}{k!l!}\sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma)\eta(v_{\sigma(k+1)}, ..., v_{\sigma(k+l)})\omega(v_{\sigma(1)}, ..., v_{\sigma(k)}) \\
&= \frac{1}{k!l!}\sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma)(\eta \otimes \omega)(\sigma\tau(v_1, ..., v_{k+l})) \\
&= \frac{(-1)^{kl}}{k!l!}\sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma\tau)(\eta \otimes \omega)(\sigma\tau(v_1, ..., v_{k+l})) \\
&= (-1)^{kl}\frac{(k+l)!}{k!l!}\text{Alt}(\eta \otimes \omega)(v_1, ..., v_{k+l})
\end{aligned}$$

and hence this proves that

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$$

and this completes the proof of (4). (5) can also be proven from first principles and I am skipping the proof (I may want to complete this proof sometime later.) ■

Theorem 8.7. *The following hold.*

(1) *If* $S \in \mathscr{T}^k(V)$ *and* $T \in \mathscr{T}^l(V)$ *and* $\text{Alt}(S) = 0$ *then*

$$\text{Alt}(S \otimes T) = \text{Alt}(T \otimes S) = 0$$

(2) $\text{Alt}(\text{Alt}(\omega \otimes \eta) \otimes \theta) = \text{Alt}(\omega \otimes \eta \otimes \theta) = \text{Alt}(\omega \otimes \text{Alt}(\eta \otimes \theta))$.

(3) *If* $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$ *and* $\theta \in \Lambda^m(V)$ *then*

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!l!m!}\mathrm{Alt}(\omega \otimes \eta \otimes \theta)$$

Proof. Let us prove (1) first. We have the following. Let G be the subgroup of S_{k+l} of all permutations that leave $\{k+1, ..., k+l\}$ *fixed* (so that $G \cong S_k$). Now, all *right cosets* of G partition S_{k+l} into disjoint subsets. So, the sum

$$\mathrm{Alt}(S \otimes T)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) = \frac{1}{(k+l)!}\sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)(S \otimes T)(v_1, ..., v_{k+l})$$

can be split into sums over the *right cosets* of G. So, it is enough to show that the sum over any *right coset* is zero. So, let $G\sigma_0$ be any right coset. First consider the case when $\sigma_0 \in G$. In that case, we see that $G\sigma_0 = G$. Now, observe that

$$\begin{aligned}\frac{1}{(k+l)!}\sum_{\sigma \in G\sigma_0} \mathrm{sgn}(\sigma)(S \otimes T)(v_1, ..., v_{k+l}) &= \frac{1}{(k+l)!}\sum_{\sigma \in G}\mathrm{sgn}(\sigma)(S \otimes T)(v_1, ..., v_{k+l}) \\ &= \frac{T(v_{k+1}, ..., v_{k+l})}{(k+l)!}\sum_{\sigma \in G}\mathrm{sgn}(\sigma)S(v_{\sigma(1)}, ..., v_{\sigma(k)}) \\ &= 0\end{aligned}$$

because we assumed that $\mathrm{Alt}(S) = 0$. In the second case, suppose $\sigma_0 \notin G$ and consider the coset $G\sigma_0$, which is not equal to G in this case. Now suppose $(v_1, ..., v_{k+l}) \in V^{k+l}$, and let

$$(v_{\sigma_0(1)}, ..., v_{\sigma_0(k+l)}) = (w_1, ..., w_{k+l})$$

So, we have

$$\begin{aligned}\frac{1}{(k+l)!}\sum_{\sigma \in G\sigma_0} \mathrm{sgn}(\sigma)(S \otimes T)(v_1, ..., v_{k+l}) &= \frac{\mathrm{sgn}(\sigma_0)}{(k+l)!}\sum_{\sigma \in G}\mathrm{sgn}(\sigma)(S \otimes T)(w_{\sigma(1)}, ..., w_{\sigma(k+l)}) \\ &= \frac{\mathrm{sgn}(\sigma_0)T(w_{k+1}, ..., w_{k+l})}{(k+l)!}\sum_{\sigma \in G}\mathrm{sgn}(\sigma)S(w_{\sigma(1)}, ..., w_{\sigma(k)}) \\ &= 0\end{aligned}$$

Hence, we conclude that $\mathrm{Alt}(S \otimes T) = 0$. By taking permutations which fix the set $\{1, ..., l\}$, we can similarly show that $\mathrm{Alt}(T \otimes S) = 0$. This completes the proof of (1).

Next, we prove (2). Observe that

$$\mathrm{Alt}[\mathrm{Alt}(\omega \otimes \eta) - \omega \otimes \eta]$$

which follows by part (3) of **Theorem** 8.5 and the fact that Alt itself is a linear map. So by (1) of the current theorem, we see that

$$\mathrm{Alt}[(\mathrm{Alt}(\omega \otimes \eta) - \omega \otimes \eta) \otimes \theta] = 0 = \mathrm{Alt}[\mathrm{Alt}(\omega \otimes \eta) \otimes \theta - \omega \otimes \eta \otimes \theta]$$

where above we have just used the distributivity of the tensor product in **Proposition** 8.1. Again using the fact that Alt itself is a linear map, we see that

$$\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \theta) = \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$$

and similarly, the other equality in (2) can be proven. This completes the proof of (2).

Finally we prove (3). We will be using properties of the wedge product proven in **Proposition** 8.6.

$$\begin{aligned}(\omega\wedge\eta)\wedge\theta &= \frac{(k+l)!}{k!l!}\mathsf{Alt}(\omega\otimes\eta)\wedge\theta\\ &= \frac{(k+l)!}{k!l!}\frac{(k+l+m)!}{(k+l)!m!}\mathsf{Alt}((\omega\otimes\eta)\otimes\theta)\\ &= \frac{(k+l+m)!}{k!l!m!}\mathsf{Alt}(\omega\otimes\eta\otimes\theta)\end{aligned}$$

where in the last step we used (2). This completes the proof. ■

Remark 8.7.1. The main point of this theorem is that wedge products are *associative*. This, as we see by the proof, is not a trivial fact.

Theorem 8.8. *Let V be any $\mathbb{R}$-vector space with basis $v_1, ..., v_n$. Let $\varphi_1, ..., \varphi_n$ be the corresponding dual basis. Then, the set of all*

$$\varphi_{i_1}\wedge\varphi_{i_2}\wedge ... \wedge\varphi_{i_k} \quad , \quad 1\le i_1 < i_2 < ... < i_k \le n$$

is a basis for $\Lambda^k(V)$, and hence

$$\dim\Lambda^k(V) = \frac{n!}{k!(n-k)!}$$

Proof. Let $\omega\in\Lambda^k(V)$. As we saw in **Theorem** 8.2, we can write

$$\omega = \sum_{(i_1,...,i_k)\in\{1,...,n\}^k} \omega(v_{i_1},...,v_{i_k})\varphi_{i_1}\otimes ... \otimes\varphi_{i_k}$$

Applying the Alt operator to both sides, we see that

$$\omega = \mathsf{Alt}(\omega) = \sum_{(i_1,...,i_k)\in\{1,...,n\}^k} \omega(v_{i_1},...,v_{i_k})\mathsf{Alt}(\varphi_{i_1}\otimes ... \otimes\varphi_{i_k})$$

Now, each $\mathsf{Alt}(\varphi_{i_1}\otimes ... \otimes\varphi_{i_k})$ is some constant times $\varphi_{i_1}\wedge ... \wedge\varphi_{i_k}$, where $i_1 < i_2 < ... < i_k$ which follows from (4) of **Proposition** 8.6 and (3) of **Theorem** 8.7. Linear independence is straightforward to prove, and hence the claim follows. ■

Theorem 8.9. *Let $v_1, ..., v_n$ be a basis for V, and let $\omega\in\Lambda^n(V)$. If $w_i = \sum_{j=1}^n a_{ij}v_j$ are n vectors in V, then*

$$\omega(w_1, .., w_n) = \mathsf{det}(a_{ij})\cdot\omega(v_1, ..., v_n)$$

Remark 8.9.1. This can be looked at in a different way. If V is an n-dimensional over $\mathbb{R}$, then **Theorem** 8.8 shows that $\Lambda^n(V)$ is a 1-dimensional space, and hence it is spanned by any non-zero element. The determinant is such a function in $\Lambda^n(\mathbb{R}^n)$, and hence we find it in this formula.

Proof. To be completed. ■

Definition 8.8. Let $\omega\in\Lambda^n(V)$, where V is an n-dimensional vector space over $\mathbb{R}$. Then, ω splits the bases of V into two equivalence classes; for a basis $\{v_1, ..., v_n\}$ of V, its class is determined by the sign of $\omega(v_1, ..., v_n)$. Any such equivalence class is called an *orientation* of V, and the orientation of a basis $\{v_1, ..., v_n\}$ is denoted by $[v_1\ v_2\ ...\ v_n]$. The *standard orientation* of $\mathbb{R}^n$ is defined as the orientation $[e_1\ e_2\ ...\ e_n]$.

Remark 8.9.2. Let V be a vector space over $\mathbb{R}$. Note that any two bases of V are related by an *invertible matrix*, the so called *change-of-basis matrix*. So we can define the orientation of a basis *independent* of any $\omega \in \Lambda^n(V)$. By **Theorem** 8.9, note that two bases have the same orientation if and only if the matrix that relates them has positive determinant. We can take this as an equivalent definition of orientation.

8.3. **Vector Fields, 1-Forms and Differential Forms.** For simplicity we will assume that all maps in consideration are $\mathscr{C}^\infty$. We define $\Lambda^0(V) := \mathbb{R}$, $\Lambda^1(V) := V^*$ and $\Lambda^k(V)$ is defined as usual.

Definition 8.9. Let V be a real vector space of dimension n, and let U be an open subset. A *vector field* is a $\mathscr{C}^\infty$ map $F : U \to V$. A 0-*form* on U is a map from U to $\Lambda^0(V) = \mathbb{R}$, i.e a real valued function. A k-*form* is a map $\omega : U \to \Lambda^k(V)$; the *degree* of such an ω is k.

Definition 8.10. Let $\{e_1, ..., e_n\}$ be a basis for V and let $\{\varphi_1, ..., \varphi_n\}$ be the corresponding dual basis of V^*. The coordinate functions $x^1, ..., x^n$ are defined by

$$x^i = \varphi_i$$

In particular, these functions are 0-forms s.t $x^i(p)$ is the i^{th} coordinate of p (where coordinates are taken w.r.t $\{e_1, ..., e_n\}$).

Definition 8.11. Let V and U be as above. Let ω be a k-form, and let η be an l-form. Define a $k+l$-form $\omega \wedge \eta$ by

$$(\omega \wedge \eta)(p) = \omega(p) \wedge \eta(p)$$

for any $p \in U$.

Definition 8.12. Let f be a function on U. Define the *exterior derivative* df to be the 1-form defined by

$$df(p)[u] = Df(p)[u]$$

where $Df(p)$ is the derivative of f at p. So, $df(p)[u]$ is the directional derivative of f at p along the vector u. Note that df is a 1-form because it maps every $p \in U$ to some linear map in V^*, namely the derivative of f at p (which we know is a linear map).

Proposition 8.10. *Let x^i be the i^{th} coordinate function. Then*

$$dx^i(p) = \varphi_i$$

for any $p \in U$. So, dx^i is a constant 1-form taking the value φ_i at all $p \in U$.

Proof. Let $p \in U$. By definition, we know that

$$dx^i(p)[u] = Dx^i(p)[u] \quad , \quad u \in V$$

Now observe that x^i is a linear map, and hence it is its own derivative. So this means that $Dx^i(p) = x^i$ for all $p \in U$. So, it follows that

$$dx^i(p) = x^i = \varphi_i$$

for all $p \in U$. ■

Proposition 8.11. *Let U, V be as in* **Definition** 8.9*, and let f be a $\mathscr{C}^\infty$ function (equivalently a 0-form) on U. Then for any $p \in U$ and $u \in V$*

$$df(p)[u] = \sum_i \frac{\partial f}{\partial x^i}(p)dx^i(p)[u] = \sum_i \frac{\partial f}{\partial x^i}\varphi_i[u]$$

The above equation is written classically as

$$df = \sum_i \frac{\partial f}{\partial x^i}dx^i$$

Observe that each $\dfrac{\partial f}{\partial x^i}$ is a 0-form and each dx^i is a 1-form.

Proof. We know that

$$df(p)[u] = Df(p)[u] = \sum_i \frac{\partial f}{\partial x^i}(p)\varphi_i[u]$$

because $Df(p)$ is a $1 \times n$ matrix which is the row vector $\nabla f(p)$. The rest of the claim is immediate. ■

Corollary 8.11.1. *If f is a $\mathscr{C}^1$ map on U (equivalently a 0-form), then df is also $\mathscr{C}^\infty$.*

Proof. df is the sum

$$df = \sum_i \frac{\partial f}{\partial x^i}\varphi_i$$

and every term in the sum is a $\mathscr{C}^\infty$ function times a constant 1-form. ■

Definition 8.13. We can extend the notion of the *exterior derivative* k-forms on U, i.e for any k-form ω, we can define its *differential* $d\omega$ which will be a $k+1$-form such that the following are satisfied.

(1) $d(df) = 0$ for any function f.
(2) Given a k-form ω and an l-form η,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

Let ω be any k-form on U. Then $\omega(p) \in \Lambda^k(V)$ for every $p \in U$, and hence by **Theorem** 8.8 ω can be uniquely written in the form

$$\omega(p) = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \omega_{i_1,\ldots,i_k}(p)dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

where $\omega_{i_1,\ldots,i_k}(p) \in \mathbb{R}$. Regarding $\omega_{i_1,\ldots,i_k}$ as a 0-form on U, we see that

$$\omega = \sum_{1 \le i_1 < \ldots < i_k \le n} \omega_{i_1,\ldots,i_k}dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

So, (1) and (2) together force the formula

$$\begin{aligned} d\omega &= \sum_{1 \le i_1 < \ldots < i_k \le n} d\omega_{i_1,\ldots,i_k}dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\ &= \sum_{1 \le i_1 < \ldots < i_k \le n, 1 \le j \le n} \frac{\partial \omega_{i_1,\ldots,i_k}}{\partial x^j}dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \end{aligned}$$

So, we can *define* the *differential* $d\omega$ by the above formula, and it turs out that this definition satisfies properties (1) and (2) above.

Proposition 8.12. *We have*

$$d(\omega + \eta) = d(\omega) + d(\eta)$$

for any k-forms ω, η. Moreover, for any k-form ω, $d(d\omega) = 0$. Briefly, this is written as $d^2 = 0$.

Proof. The first formula is immediate. If

$$\omega = \sum_{1 \leq i_1 < ... < i_k \leq n} \omega_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}$$

then we have

$$d\omega = \sum_{1 \leq i_1 < ... < i_k \leq n, 1 \leq j \leq n} \frac{\partial \omega_{i_1,...,i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}$$

So, we have that

$$d(d\omega) = \sum_{1 \leq i_1 < ... < i_k \leq n, 1 \leq j \leq n, 1 \leq m \leq n} \frac{\partial^2 \omega_{i_1,...,i_k}}{\partial x^m \partial x^j} dx^m \wedge dx^j \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}$$

Now in the above formula, terms of the form

$$\frac{\partial^2 \omega_{i_1,...,i_k}}{\partial x^m \partial x^j} dx^m \wedge dx^j \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}, \frac{\partial^2 \omega_{i_1,...,i_k}}{\partial x^j \partial x^m} dx^j \wedge dx^m \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}$$

will cancel pairwise. This completes the proof. ∎

8.4. **Pull-Backs.** Suppose we are given a $\mathscr{C}^\infty$ map $\Phi : \tilde{U} \to U$ of open sets, with $U \subset V$, $\tilde{U} \subset \tilde{V}$ where $V, \tilde{V}$ are finite dimensional vector spaces of dimension n and m respectively. Given a point $\tilde{p} \in \tilde{U}$, the differential $D\Phi(\tilde{p})$ gives a linear map $\tilde{V} \to V$. Using Φ, given any $\mathscr{C}^\infty$ function on U, a $\mathscr{C}^\infty$ on $\tilde{U}$ can be defined by simply composing with Φ. We call this map Φ_0^*, which is defined as follows.

$$\Phi_0^* : \{\mathscr{C}^\infty \text{ functions on } U\} \to \{\mathscr{C}^\infty \text{ functions on } \tilde{U}\}$$
$$f \mapsto \Phi_0^*(f) = f \circ \Phi$$

This can be extended to k-forms, and we will call the corresponding map Φ_k^* and we will extend it so that this new map satisfies two additional properties. So

$$\Phi_k^* : \{\mathscr{C}^\infty\ k \text{ forms on } U\} \to \{\mathscr{C}^\infty\ k \text{ forms on } \tilde{U}\}$$
$$\eta \mapsto \Phi_k^*(\eta)$$

is a map which satisfies the following two properties:

$$\Phi_{k+1}^* d\omega = d\Phi_k^* \omega \tag{†}$$

$$\Phi_{k+l}^* (\omega \wedge \eta) = \Phi_k^* \omega \wedge \Phi_l^* \eta \tag{‡}$$

From here on, we will omit the subscript k and simply write Φ^* in place of Φ_k^*.

Proposition 8.13. *Let $\Phi : \tilde{U} \to U$ be as above. Let bases of V and $\tilde{V}$ be fixed, and suppose $x^1, ..., x^n$ and $t^1, ..., t^m$ are the respective coordinate functions. Then*

$$\Phi^* dx^i = \sum_{j=1}^m \frac{\partial (x^i \circ \Phi)}{\partial t^j} dt^j$$

Proof. By property (†), we have

$$\Phi^* dx^i = d\Phi^* x^i = d(x^i \circ \Phi)$$

where $\Phi^* x^i = x^i \circ \Phi$ because x^i is a 0-form (equivalently a $\mathscr{C}^\infty$ function). By **Proposition** 8.11, we know that

$$d(x^i \circ \Phi) = \sum_{j=1}^{m} \frac{\partial(x^i \circ \Phi)}{\partial t^i} dt^i$$

■

Proposition 8.14. *Let η be a 1-form on U. Let $\tilde{p} \in \tilde{U}$, and let $p = \Phi(\tilde{p}) \in U$. Then the following equation relates $\Phi^*\eta(p)$ and $\eta(p)$:*

$$\Phi^*\eta(\tilde{p})[u] = \eta(p)\left[D\Phi(\tilde{p})[u]\right]$$

Proof. Again, let bases $x_1, ..., x_n$ and $t_1, ..., t_m$ of V and $\tilde{V}$ be fixed, and let $x^1, ..., x^n$ and $t^1, ..., t^m$ be the respective coordinate functions. Since η is a 1-form on U, we can write

$$\eta = \sum_{i=1}^{n} \eta_i dx^i$$

where each η_i is a function (equivalently a 0-form) on U. So by the *additivity* (need to prove this! Update: it turns out that this must be assumed as a property of Φ^*) of Φ^*, we see that

$$\Phi^*\eta = \sum_{i=1}^{n} \Phi^*(\eta_i dx^i)$$

Writing $\eta_i dx^i = \eta_i \wedge dx^i$, we see that $\Phi^*(\eta_i dx^i) = \Phi^*\eta_i \wedge \Phi^* dx^i$. Now η_i is a function and $\Phi^* dx^i$ can be computed as in **Proposition** 8.13. So, we have

$$\Phi^*\eta_i \wedge \Phi^* dx^i = (\eta_i \circ \Phi) \wedge \sum_{j=1}^{m} \frac{\partial(x^i \circ \Phi)}{\partial t^j} dt^j = \sum_{j=1}^{m} (\eta_i \circ \Phi) \frac{\partial(x^i \circ \Phi)}{\partial t^j} dt^j$$

So, we have

$$\Phi^*\eta = \sum_{i=1}^{n} \sum_{j=1}^{m} (\eta_i \circ \Phi) \frac{\partial(x^i \circ \Phi)}{\partial t^j} dt^j$$

The above equation means that for any $\tilde{p} \in \tilde{U}$, $u \in \tilde{V}$ and $p = \Phi(p)$,

$$\begin{aligned}
\Phi^*\eta(\tilde{p})[u] &= \sum_{i=1}^{n} \sum_{j=1}^{m} (\eta_i \circ \Phi)(\tilde{p}) \frac{\partial(x^i \circ \Phi)(\tilde{p})}{\partial t^j} dt^j(\tilde{p})[u] \\
&= \sum_{i=1}^{n} \eta_i(p) \sum_{j=1}^{m} \frac{\partial(x^i \circ \Phi)(\tilde{p})}{\partial t^j} t^j[u] \\
&= \sum_{i=1}^{n} \eta_i(p) x^i[D\Phi(\tilde{p})[u]] \\
&= \sum_{i=1}^{n} \eta_i(p) dx^i(p)[D\Phi(\tilde{p})[u]] \\
&= \eta(p)[D\Phi(\tilde{p})[u]]
\end{aligned}$$

and this proves the claim. ■

Proposition 8.15. *Let $m = n$, and let $x_1, ..., x_n$ be a basis for V and let $t_1, ..., t_n$ be a basis for $\tilde{V}$. Then*

$$\Phi^* dx^1 \wedge ... \wedge dx^n = J_\Phi dt^1 \wedge ... \wedge dt^n$$

where J_Φ is the determinant of the Jacobian of Φ.

Proof. This is a rather straightforward computation.

$$\begin{aligned}
\Phi^* dx^1 \wedge ... \wedge dx^n &= \Phi^* dx^1 \wedge ... \wedge \Phi^* dx^n \\
&= d\Phi^* x^1 \wedge ... \wedge d\Phi^* x^n \\
&= d(x^1 \circ \Phi) \wedge ... \wedge d(x^n \circ \Phi) \\
&= \left(\sum_{i=1}^n \frac{\partial (x^1 \circ \Phi)}{\partial t^i} dt^i \right) \wedge ... \wedge \left(\sum_{i=1}^n \frac{\partial (x^n \circ \Phi)}{\partial t^i} dt^i \right) \\
&= \sum_{i_1=1}^n ... \sum_{i_n=1}^n \frac{\partial (x^1 \circ \Phi)}{\partial t^{i_1}} ... \frac{\partial (x^n \circ \Phi)}{\partial t^{i_n}} dt^{i_1} \wedge ... \wedge dt^{i_n} \\
&= \sum_{\sigma \in S_n} \frac{\partial (x^1 \circ \Phi)}{\partial t^{\sigma(1)}} ... \frac{\partial (x^n \circ \Phi)}{\partial t^{\sigma(n)}} \text{sgn}(\sigma) dt^1 \wedge ... \wedge dt^n \\
&= J_\Phi dt^1 \wedge ... \wedge dt^n
\end{aligned}$$

Above, we used two crucial properties of wedge products. ■

Example 8.1. (Exterior Calculus in $\mathbb{R}^3$). Let x, y, z denote the standard coordinate functions in $\mathbb{R}^3$. We will investigate the formulae for differentials in this setting.

(1) Given a function f,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

(2) Given a 1-form $\omega = \omega_x dx + \omega_y dy + \omega_z dz$ where each $\omega_x, \omega_y, \omega_z$ is a function, we have the following.

$$\begin{aligned}
d\omega &= d\omega_x \wedge dx + d\omega_y \wedge dy + d\omega_z \wedge dz \\
&= \left(\frac{\partial \omega_x}{\partial x} dx \wedge dx + \frac{\partial \omega_x}{\partial y} dy \wedge dx + \frac{\partial \omega_x}{\partial z} dz \wedge dx \right) + \\
&\quad \left(\frac{\partial \omega_y}{\partial x} dx \wedge dy + \frac{\partial \omega_y}{\partial y} dy \wedge dy + \frac{\partial \omega_y}{\partial z} dz \wedge dy \right) + \\
&\quad \left(\frac{\partial \omega_z}{\partial x} dx \wedge dz + \frac{\partial \omega_z}{\partial y} dy \wedge dz + \frac{\partial \omega_z}{\partial z} dz \wedge dz \right) \\
&= \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) dx \wedge dy + \left(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z} \right) dy \wedge dz + \left(\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x} \right) dz \wedge dx
\end{aligned}$$

(3) Given a 2-form $\eta = \eta_{xy} dx \wedge dy + \eta_{yz} dy \wedge dz + \eta_{zx} dz \wedge dx$ we have the following.

$$\begin{aligned} d\eta &= d\eta_{xy} \wedge dx \wedge dy + d\eta_{yz} \wedge dy \wedge dz + d\eta_{zx} \wedge dz \wedge dx \\ &= \left(\frac{\partial \eta_{xy}}{\partial x} dx + \frac{\partial \eta_{xy}}{\partial y} dy + \frac{\partial \eta_{xy}}{\partial z} dz \right) \wedge dx \wedge dy + \\ &\left(\frac{\partial \eta_{yz}}{\partial x} dx + \frac{\partial \eta_{yz}}{\partial y} dy + \frac{\partial \eta_{yz}}{\partial z} dz \right) \wedge dy \wedge dz + \\ &\left(\frac{\partial \eta_{zx}}{\partial x} dx + \frac{\partial \eta_{zx}}{\partial y} dy + \frac{\partial \eta_{zx}}{\partial z} dz \right) \wedge dz \wedge dx \\ &= \left(\frac{\partial \eta_{xy}}{\partial z} + \frac{\partial \eta_{yz}}{\partial x} + \frac{\partial \eta_{zx}}{\partial y} \right) dx \wedge dy \wedge dz \end{aligned}$$

(4) If we are given a 1-form $\rho = \rho_{xyz} dx \wedge dy \wedge dz$, then we easily see that $d\rho = 0$.

Example 8.2. (Exterior Calculus in $\mathbb{R}^2$). Let x, y denote the standard coordinate functions in $\mathbb{R}^2$. Here we will investigate the formulae for differentials.

(1) Given a function f, we have

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

(2) Given a 1-form $\omega = \omega_x dx + \omega_y dy$, we have the following.

$$\begin{aligned} d\omega &= d\omega_x \wedge dx + d\omega_y \wedge dy \\ &= \left(\frac{\partial \omega_x}{\partial x} dx + \frac{\partial \omega_x}{\partial y} dy \right) \wedge dx + \left(\frac{\partial \omega_y}{\partial x} dx + \frac{\partial \omega_y}{\partial y} dy \right) \wedge dy \\ &= \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) dx \wedge dy \end{aligned}$$

(3) If we are given a 2-form $\eta = \eta_{xy} dx \wedge dy$ then we can easily see that $d\eta = 0$.

Theorem 8.16. *Let ω be any 1-form in $\mathbb{R}^2$ such that $d\omega = 0$. Then, there is a function f such that $df = \omega$.*

Remark 8.16.1. 1-forms ω for which $d\omega = 0$ are called *closed forms*. Those ω for which $\omega = d\eta$ for some function η are called *exact forms*.

Proof. Let ω be a 1-form on $\mathbb{R}^2$ where

$$\omega = \omega_1 dx + \omega_2 dy$$

such that $d\omega = 0$, i.e ω is a closed 1-form (here ω_1, ω_2 are $\mathscr{C}^\infty$ real valued maps on $\mathbb{R}^2$). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = \int_0^1 (x\omega_1(xt, yt) + y\omega_2(xt, yt))dt$$

By Leibniz' Rule, we have that

$$D_1 f(x, y) = \int_0^1 \omega_1(xt, yt) + xtD_1\omega_1(xt, yt) + ytD_1\omega_2(xt, yt)dt$$

Because ω is a closed form, we have

$$D_1\omega_2 = D_2\omega_1$$

So, we see that

$$\begin{aligned} D_1 f(x,y) &= \int_0^1 \omega_1(xt,yt) + xtD_1\omega_1(xt,yt) + ytD_1\omega_2(xt,yt)dt \\ &= \int_0^1 \omega_1(xt,yt) + xtD_1\omega_1(xt,yt) + ytD_2\omega_1(xt,yt)dt \\ &= \int_0^1 \frac{d}{dt}[t\omega_1(xt,yt)]dt \\ &= t\omega_1(xt,yt)|_0^1 \\ &= \omega_1(x,y) \end{aligned}$$

where in the last step we used the fundamental theorem of calculus. Similarly, we can show that

$$D_2 f(x,y) = \omega_2(x,y)$$

and hence it follows that $\omega = df$, completing the proof. ■

Remark 8.16.2. Observe that the same proof will work if ω was a 1-form on an open set U, where U is *star shaped*, i.e $0 \in U$ and for any $(x,y) \in U$, the line $t(x,y)$ for $0 \le t \le 1$ is contained in U.

8.5. **Singular Cubical Chains.** Let S be any set. The free $\mathbb{Z}$-module over the set S is the set of all maps $n : S \to \mathbb{Z}$ such that $n(c) \neq 0$ for only finitely many $c \in S$. This set clearly is a $\mathbb{Z}$-module with a basis, and hence it is free. Any element n of this module is written as

$$\sum_{c \in S} n(c)c$$

where the above sum only includes those $c \in S$ for which $n(c) \neq 0$.

Definition 8.14. Let A be any topological space. A *singular n-cube in A* is a continuous map $c : [0,1]^n \to A$. Most of the time A will be an open subset of $\mathbb{R}^m$ for some $m \in \mathbb{N}$.

A *singular cubical n-chain* in A is an element of the free $\mathbb{Z}$-module generated by n-cubes in A. Informally, it is a finite sum

$$\sum_i n_i c_i$$

where $n_i \in \mathbb{Z}$ and c_i is an n-cube in A.

Example 8.3. A 0-cube is a map $c : \{0\} \to A$, and we identify it with its image. So, a 0-chain is just the free $\mathbb{Z}$-module generated by the set.

Definition 8.15. For an n-cube c, the *boundary* of the cube, denoted by ∂c, is defined as an $(n-1)$-chain whose definition we will see below for $n = 1, 2$. Once this is defined, the boundary of an n-chain is defined by linearity as

$$\partial \left(\sum_i n_i c_i \right) := \sum_i n_i \partial c_i$$

(1) For a 1-cube $\gamma : [0,1] \to A$, we define

$$\partial \gamma = \gamma(1) - \gamma(0)$$

where the RHS is an element of the free $\mathbb{Z}$-module generated by A.

(2) Consider the identity 1-cube $I^2 : [0,1]^2 \to [0,1]^2$, i.e I^2 is the identity map. We define ∂I^2 to be the four edges of the cube in $\mathbb{R}^2$ in counter-clockwise orientation, i.e

$$\partial I^2 = -I^2_{(1,0)} + I^2_{(1,1)} + I^2_{(2,0)} - I^2_{(2,1)}$$

where each term on the RHS is a 1-cube, each mapping the interval $[0,1]$ to an edge of the square $[0,1]^2$.

$$\begin{aligned} I^2_{(1,0)}(t) &= (0,t) \\ I^2_{(1,1)}(t) &= (1,t) \\ I^2_{(2,0)}(t) &= (t,0) \\ I^2_{(2,1)}(t) &= (t,1) \end{aligned}$$

The boundary ∂c of an arbitrary 2-cube $c : [0,1]^2 \to A$ is defined as

$$\partial c = c \circ \partial I^2 := -c \circ I^2_{(1,0)} + c \circ I^2_{(1,1)} + c \circ I^2_{(2,0)} - c \circ I^2_{(2,1)}$$

In general, once we know that definition of ∂I^n, we can define ∂c for any n-cube c by $\partial c = c \circ \partial I^n$.

Proposition 8.17. *For any 2-cube c, $\partial(\partial c) = 0$.*

Proof. Let $c : [0,1]^2 \to [0,1]^2$ be any arbitrary 2-cube. Then, we have the following.

$$\begin{aligned} \partial(\partial c) &= \partial(-c \circ I^2_{(1,0)} + c \circ I^2_{(1,1)} + c \circ I^2_{(2,0)} - c \circ I^2_{(2,1)}) \\ &= -\partial(c \circ I^2_{(1,0)}) + \partial(c \circ I^2_{(1,1)}) + \partial(c \circ I^2_{(2,0)}) - \partial(c \circ I^2_{(2,1)}) \\ &= -(c(0,1) - c(0,0)) + (c(1,1) - c(1,0)) + (c(1,0) - c(0,0)) - (c(1,1) - c(0,1)) \\ &= 0 \end{aligned}$$

and this completes the proof. ∎

Remark 8.17.1. This fact is true for general n-cubes as well.

8.6. **Stokes Theorem in One Dimension.** Let's start with a discussion of this theorem in one dimension. Let $\tilde{\omega}$ be a 1-form defined on an open interval I containing $[0,1]$. Let t denote the natural coordinate in $\mathbb{R}$. So, $\tilde{\omega} = \tilde{\omega}_t dt$ for some function $\tilde{\omega}_t$ on I. We define

$$\int_{[0,1]} \omega := \int_0^1 \tilde{\omega}(t)dt$$

where the integral on the RHS is the ordinary Riemann Integral.

Let ω be a $\mathscr{C}^\infty$ 1-form on an open set U in $\mathbb{R}$. Consider a singular 1-cube, i.e a $\mathscr{C}^\infty$ map $\gamma : [0,1] \to U$. To say that γ is $\mathscr{C}^\infty$ just means that γ extends as a $\mathscr{C}^\infty$ map on an open interval $\tilde{I}$ containing $[0,1]$. Let x denote the natural coordinate function on U and let t be the natural coordinate function on $\tilde{I}$. Then we have the following.

(1) $\omega = \omega_x dx$ for a unique function ω_x on U.
(2) $\gamma^*\omega = \gamma^*(\omega_x dx) = \gamma^*\omega_x \wedge d\gamma^* x = (\omega_x \circ \gamma)\dfrac{d(x \circ \gamma)}{dt}dt$

So we define

$$\int_\gamma \omega := \int_{[0,1]} \gamma^*\omega = \int_0^1 (\omega_x \circ \gamma)(t)\frac{d(x\circ\gamma)(t)}{dt}dt$$

Now, suppose $\omega = df$ for some function f on U. Then,

$$\int_\gamma df = \int_0^1 \frac{df}{dt}(\gamma(t))\frac{d\gamma(t)}{dt}dt = f(\gamma(1)) - f(\gamma(0))$$

Observe that $\partial\gamma$ is the 0-chain $\gamma(1) - \gamma(0)$. So, the above equation can be written as

$$\int_\gamma df = \int_{\partial\gamma} f$$

For a 1-chain, we define

$$\int_{\sum_i n_i\gamma_i} \omega := \sum_i n_i \int_{\gamma_i} \omega = \sum_i n_i \int_{[0,1]} \gamma^*\omega$$

and so we see that for a function f on U,

$$\int_{\sum_i n_i\gamma_i} df = \sum_i n_i \int_{\gamma_i} df = \sum_i n_i \int_{\partial\gamma_i} f =: \int_{\partial(\sum_i n_i\gamma_i)} f$$

8.7. **Stokes Theorem with proof in dimension two.** The general version of **Stokes' Theorem** is as follows.

Theorem 8.18 (**Stokes' Theorem**). *Let η be a $k-1$-form on an open set $U \subset V$, where V is an n-dimensional real vector space. Let $\sum_i n_i c_i$ be a k-chain in U. Then*

$$\int_{\sum_{n_i c_i}} d\eta = \int_{\partial(\sum_i n_i c_i)} \eta$$

Proof. Proof for $k = 1$ is above, and for $k = 2$ it is given in the lecture notes. ∎

Example 8.4. Here is an interesting example. Consider the 1-form η on $\mathbb{R}^2 \setminus \{(0,0)\}$.

$$\eta = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

(1) Let us first compute $d\eta$. We have

$$d\eta = \left(\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right) dx \wedge dy = 0$$

(2) Let $\Phi : \{(r,\theta) \mid r > 0\} \to \mathbb{R}^2 \setminus \{(0,0)\}$ be the map

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$$

We show that $\Phi^*\eta = d\theta$. We have the following chain of equations.

$$\begin{aligned}
\Phi^*\eta &= \Phi^*\left(\frac{-y}{x^2+y^2}\right) \wedge \Phi^* dx + \Phi^*\left(\frac{x}{x^2+y^2}\right) \wedge \Phi^* dy \\
&= \left(\frac{-y}{x^2+y^2}\right) \circ \Phi \wedge d(x \circ \Phi) + \left(\frac{x}{x^2+y^2}\right) \circ \Phi \wedge d(y \circ \Phi) \\
&= \left(\frac{-r\sin\theta}{r^2}\right) \wedge d(r\cos\theta) + \left(\frac{r\cos\theta}{r^2}\right) \wedge d(r\sin\theta) \\
&= \frac{-r\sin\theta}{r^2}[\cos\theta dr - r\sin\theta d\theta] + \frac{r\cos\theta}{r^2}[\sin\theta dr + r\cos\theta d\theta] \\
&= d\theta
\end{aligned}$$

(3) For $n \in \mathbb{Z}$, let γ_n be the 1-cube in $\mathbb{R}^2 \setminus \{(0,0)\}$ given by

$$\gamma_n(t) = (\cos 2\pi nt, \sin 2\pi nt)$$

We show that

$$\int_\gamma \eta = 2\pi n$$

Observe that

$$\int_\gamma \eta = \int_{[0,1]} \gamma^*\eta$$

Now,

$$\begin{aligned}
\gamma^*\eta &= \gamma^*\left(\frac{-y}{x^2+y^2}\right) \wedge \gamma^* dx + \gamma^*\left(\frac{x}{x^2+y^2}\right) \wedge \gamma^* dy \\
&= -\sin 2\pi nt \wedge d(x \circ \gamma) + \cos 2\pi nt \wedge d(y \circ \gamma) \\
&= -\sin 2\pi nt \wedge d(\cos 2\pi nt) + \cos 2\pi nt \wedge d(\sin 2\pi nt) \\
&= (2\pi n)\sin^2 2\pi nt dt + (2\pi n)\cos^2 2\pi nt dt \\
&= 2\pi n dt
\end{aligned}$$

and hence

$$\int_\gamma \eta = \int_0^1 2\pi n dt = 2\pi n$$

(4) So, we see that there is no 2-chain c in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $\partial c = \eta$, and this is an easy consequence of **Stokes Theorem** 8.18 because we have already computed that $d\eta = 0$.

(5) From this, we can also conclude that there is no function θ such that $d\theta = \eta$. To see this, if there was such a function, then we would get by **Stokes Theorem** 8.18

$$\int_\gamma \eta = \int_\gamma d\theta = \int_{\partial\gamma} \theta$$

However, because $\gamma(1) - \gamma(0) = 0$, it follows that $\partial\gamma = 0$ which implies that the integral on the extreme right side is zero, a contradiction.