COOKBOOK-1 SOLUTIONS

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1). $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1+y^2}{x}, \quad x > 0$

Solution. This is seen to be a separable DE. We have

$$\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

So, integrating both sides with respect to x, we see that

$$\int \frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int \frac{1}{1+y^2}\mathrm{d}y = \int \frac{1}{x}\mathrm{d}x = \ln\, x + K$$

for some $K \in \mathbb{R}$. Now,

$$\int \frac{1}{1+y^2}\mathrm{d}y = \arctan\, y + K'$$

for some $K' \in \mathbb{R}$. Combining all of this, we get

$$\arctan\, y = \ln\, x + C$$

for some $C \in \mathbb{R}$, and this is the general solution of the DE. ■

2). $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2+xy+y^2}{x^2}.$

Solution. This is an example of a homogeneous first order DE. Assume that $x \neq 0$ and put $y = vx$. The equation reduces to

$$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = g(v) = 1 + v + v^2$$

which is the same as the equation

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v^2$$

This equation is clearly *separable*. We get

$$\frac{1}{1+v^2}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}$$

This can be easily solved to obtain

$$\arctan v = \ln x + C$$

for some constant C. This gives us

$$\arctan\left(\frac{y}{x}\right) = \ln x + C$$

■

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3). $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3x^2 - 2x - 1}{2(y-1)}, \; y(3) = 1 - \sqrt{13}.$

Solution. This is easily seen to be a *separable* DE. In particular, we have

$$(2y-2)\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2x - 1 \tag{0.1}$$

Now, consider the function $g(y) = 2y - 2$. The antiderivative of $g(y)$ is

$$\int g(y)\mathrm{d}y = y^2 - 2y + K$$

for some $K \in \mathbb{R}$. So, integrating both sides of equation (0.2) with respect to x, we get

$$\int (2y-2)\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int (2y-2)\mathrm{d}y = \int (3x^2 - 2x - 1)\mathrm{d}x$$

and hence we obtain

$$y^2 - 2y = x^3 - x^2 - x + C$$

for some $C \in \mathbb{R}$. Using the initial condition $y(3) = 1 - \sqrt{13}$, we get

$$(1-\sqrt{13})^2 - 2(1-\sqrt{13}) = 27 - 9 - 3 + C$$

and from here we obtain $C = -3$. Hence, we obtain the implicit equation

$$y^2 - 2y - x^3 + x^2 + x + 3 = 0$$

■

4). $(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 1 + x, \quad x > 0.$

Solution. This is easily seen to be a first order linear DE. Dividing throughout by $(1+x)$ we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{(1+x)}y = 1$$

Observe that

$$\int \frac{1}{(1+x)}\mathrm{d}x = \ln(1+x) + K$$

for some constant K. So, the integrating factor is

$$\mu(x) = e^{\ln(1+x)+K} = K'(1+x)$$

where $K' \in \mathbb{R}$. So, the solution of the DE is given by

$$y = \frac{1}{K'(1+x)}\int K'(1+x)dx = \frac{1}{(1+x)}\left[\frac{(1+x)^2}{2} + R\right]$$

for some constant $R \in \mathbb{R}$. ■

5). $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x - 2x^3}{16 + 2y^3}$

Solution. This is again a *separable* DE. We get

$$(16 + 2y^3)\frac{\mathrm{d}y}{\mathrm{d}x} = x - 2x^3$$

Integrating both sides of the above equation, we can obtain

$$y^4 + 32y = x^2 - x^4 + C$$

■

6). Suppose $y = \varphi(x)$ is a solution of the problem in 5) such that $\varphi(x_0) = y_0$, and such that φ is defined as a $\mathscr{C}^1$ function in a neighborhood of x_0. We find the forbidden values of y_0.

Solution. The idea is to use the Implicit Function Theorem. Consider the function

$$\Phi(x, y) = y^4 + 32y + x^4 - x^2 - C$$

So, $(x_0, y_0) \in M$, where $M = \Phi^{-1}(0)$ and φ is a solution to the implicit equation

$$\Phi(x, \varphi(x)) = 0$$

around the point x_0. So, we want

$$\frac{\partial \Phi}{\partial y}(x_0, y_0) \neq 0$$

This is equivalent to saying

$$4y^3 + 32 \neq 0 \quad , \quad \text{at } y = y_0$$

which is equivalent to saying $y_0^3 \neq -8$, and hence $y_0 \neq -2$. ■

7). $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x - e^{-x}}{y + e^y}$

Solution. This is again a *separable* DE. We get

$$(y + e^y)\frac{\mathrm{d}y}{\mathrm{d}x} = x - e^{-x}$$

By integrating both sides, we can obtain

$$\frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + C$$

where $C \in \mathbb{R}$. ■

8). $x^2y' + 2xy - y^3 = 0, \quad x > 0.$

Solution. Dividing throughout by x^2, we obtain

$$y' + \frac{2}{x}y = \frac{1}{x^2}y^3$$

This is an example of a *Bernoulli Equation*. We substitute $v = y^{1-3} = y^{-2}$ and get

$$-\frac{1}{2}v' + \frac{2}{x}v = \frac{1}{x^2}$$

which is the same as the equation

$$v' - \frac{4}{x}v = -\frac{2}{x^2}$$

and this is a linear first order DE. We have

$$\int -\frac{4}{x}\mathrm{d}x = -4\ln x + K$$

for some $K \in \mathbb{R}$. So the integrating factor is

$$\mu(x) = e^{-4\ln x + K} = \frac{K'}{x^4}$$

where $K' \in \mathbb{R}$. So, the solution of the linear DE is

$$v = \frac{x^4}{K'}\int \frac{K'}{x^4}\cdot\frac{-2}{x^2}\mathrm{d}x = -2x^4\left(\frac{-5}{x^5} + R\right)$$

where $R \in \mathbb{R}$ is some constant. From this, y can be obtained. ■

9). $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^2 - 4x^2}{2xy}$

Solution. This is the same as the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{2x} - \frac{2x}{y}$$

This is a *homogeneous* DE. We assume $x \neq 0$, and we put $y = vx$. The equation becomes

$$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = g(v) = \frac{v}{2} - \frac{2}{v}$$

which is the same as the equation

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v}{2} - \frac{2}{v} = \frac{v^2 - 4}{2v}$$

and this is a separable DE. We get

$$\frac{2v}{v^2 - 4}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}$$

Integrating both sides with respect to x, we can solve for v, and hence we can solve for y. ■

10). $t\ln t\dfrac{\mathrm{d}u}{\mathrm{d}t} + u = te^t, \quad t > 1$

Solution. Dividing throughout by $t\ln t$, we get

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{t\ln t}u = \frac{e^t}{\ln t}$$

This is a first order linear DE. Observe that

$$\int \frac{1}{t\ln t}\mathrm{d}t = \ln\ln t + K$$

for some $K \in \mathbb{R}$. So, the integrating factor is

$$e^{\ln\ln t + K} = K'\ln t$$

where $K' \in \mathbb{R}$. So, the solution of the DE is

$$u = \frac{1}{K'\ln t}\int K'\ln t\frac{e^t}{\ln t}\mathrm{d}t = \frac{1}{\ln t}(e^t + C)$$

for some $C \in \mathbb{R}$. ■

11). $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = e^{-2x}$

Solution. This is a clear-cut first order linear DE. We have

$$\int \mathrm{d}x = x + K$$

for some $K \in \mathbb{R}$. Hence, the integrating factor is

$$\mu(x) = e^{x+K} = K'e^x$$

for some $K' \in \mathbb{R}$. So the solution of the DE is

$$y = \frac{1}{K'e^x}\int K'e^x e^{-2x}\mathrm{d}x = \frac{1}{e^x}(-e^{-x} + C)$$

for some $C \in \mathbb{R}$. ■

12). $\sin x \frac{\mathrm{d}y}{\mathrm{d}x} + (\cos x)y = e^x$

Solution. For simplicity we assume that $x \in (0, \pi)$. So, dividing throughout by the sine term, this becomes a linear first order DE.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + (\cot\ x)y = \frac{e^x}{\sin\ x}$$

So, the integrating factor $\mu(x)$ in our case is

$$\mu(x) = \exp \int \cot\ x \mathrm{d}x = \exp\left[\ln\ \sin\ x + K\right]$$

for some $K \in \mathbb{R}$. So, the integrating factor is

$$\mu(x) = e^K e^{\ln\ \sin\ x} = K' e^{\ln\ \sin\ x} = K' \sin\ x$$

So, the general solution to this DE is given by

$$y = \mu(x)^{-1} \int \mu(x) \frac{e^x}{\sin\ x} \mathrm{d}x = \frac{1}{K' \sin\ x} \int K' \sin\ x \frac{e^x}{\sin\ x} \mathrm{d}x = \frac{e^x + C}{\sin\ x}$$

for some $C \in \mathbb{R}$. ■

13). $(e^x \sin y - 2y \sin x)\mathrm{d}x + (e^x \cos y + 2\cos x)\mathrm{d}y = 0$

Solution. This equation is equivalent to

$$(e^x \sin y - 2y \sin x) + (e^x \cos y + 2\cos x)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

We now show that this is an exact equation. To see this, observe that

$$\frac{\partial(e^x \sin y - 2y \sin x)}{\partial y} = e^x \cos y - 2 \sin x$$

and that

$$\frac{\partial(e^x \cos y + 2\cos x)}{\partial x} = e^x \cos y - 2 \sin x$$

and the above two equations show that our DE is exact. Now, we need to find P such that

$$\frac{\partial P}{\partial x} = e^x \sin y - 2y \sin x \quad , \quad \frac{\partial P}{\partial y} = e^x \cos y + 2\cos x \tag{0.2}$$

So, consider

$$\begin{aligned} P(x, y) &= \int e^x \sin y - 2y \sin x \, \mathrm{d}x \\ &= e^x \sin y + 2y \cos x + g(y) \end{aligned}$$

for some differentiable function g of y. Clearly, differentiating $P(x, y)$ with respect to x, we see that $P(x, y)$ satisifies the first half of equation (0.2). Now, differentiating $P(x, y)$ with respect to y and equating it to the RHS of the second half of equation (0.2) we get

$$\frac{\partial P}{\partial y} = e^x \cos y + 2\cos x + g'(y) = e^x \cos y + 2\cos x$$

so that $g'(y) = 0$, i.e $g(y) = K$ for some $K \in \mathbb{R}$. We can take $K = 0$. So, the solution to our DE is given by the implicit equation

$$P(x, y) = e^x \sin y + 2y \cos x = C$$

for some $C \in \mathbb{R}$. ■

14). $(1+x)\frac{\mathrm{d}y}{\mathrm{d}x} + y = 1 + x, \quad x > 0$

Solution. This is the same as problem 4). ■

15). $xy' = y + x^2 \sin x, \quad y(\pi) = 0$

Solution. Dividing throughout by x, we get the equation

$$y' - \frac{1}{x}y = x \sin x$$

This is a first order linear DE. Observe that

$$\int \frac{1}{x}\,\mathrm{d}x = \ln x + K$$

for some $K \in \mathbb{R}$. So the integrating factor is

$$\mu(x) = e^{\ln x + K} = K'x$$

where $K' \in \mathbb{R}$. So, the solution of our DE is

$$y = \frac{1}{K'x}\int K'x^2 \sin x\,\mathrm{d}x = \frac{1}{x}(2x\sin x - (x^2-2)\cos x + C)$$

for some $C \in \mathbb{R}$. Using the fact that $y(\pi) = 0$ we can get the value of C. ■

16). $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = \frac{y^3}{x^2}$

Solution. This is the same as problem 8). ■

17). $y\cos x + 2xe^y + (\sin x + x^2e^y - 1)y' = 0$

Solution. This is an example of an *exact* DE. To show this, observe that

$$\frac{\partial(y\cos x + 2xe^y)}{\partial y} = \cos x + 2xe^y$$

and

$$\frac{\partial(\sin x + x^2e^y - 1)}{\partial x} = \cos x + 2xe^y$$

and hence

$$\frac{\partial(y\cos x + 2xe^y)}{\partial y} = \frac{\partial(\sin x + x^2e^y - 1)}{\partial x}$$

Now, put

$$P(x,y) = \int y\cos x + 2xe^y\,\mathrm{d}x = y\sin x + x^2e^y + g(y)$$

where g is some differentiable function of y. Now, we have

$$\frac{\partial P(x,y)}{\partial y} = \sin x + 2xe^y + g'(y)$$

Now, put

$$\sin x + 2xe^y + g'(y) = \sin x + 2xe^y - 1$$

and hence we obtain $g'(y) = -1$, which gives us $g(y) = -y + C$, for some constant C. We take $C = 0$. So,

$$P(x,y) = y\sin x + x^2e^y - y$$

So, our DE is

$$y \sin x + x^2 e^y - y = C$$

where $C \in \mathbb{R}$ is some constant. ■

18). $x^2 y'' + 2xy' - 1 = 0, \quad x > 0$

Solution. This is a second order DE with the dependent variable missing. Note that the DE is equivalent to

$$y'' = \frac{-2}{x} y' + \frac{1}{x^2}$$

Putting $v = y'$, we get the equation

$$v' = \frac{-2}{x} v + \frac{1}{x^2}$$

which is the same as the equation

$$v' + \frac{2}{x} v = \frac{1}{x^2}$$

This is a first order linear DE. Observe that

$$\int \frac{2}{x} \, \mathrm{d}x = 2 \ln x + K$$

for some $K \in \mathbb{R}$. So the integrating factor is

$$\mu(x) = e^{2 \ln x + K} = K' x^2$$

and hence the solution to the linear DE is

$$v = \frac{1}{K' x^2} \int K' x^2 \frac{1}{x^2} \, \mathrm{d}x = \frac{1}{x^2}(x + C_1)$$

for some $C_1 \in \mathbb{R}$. So, we get

$$y = \int v \, \mathrm{d}x = \int \frac{1}{x^2}(x + C_1) \, \mathrm{d}x = \ln x - \frac{C_1}{x^2} + C_2$$

where $C_2 \in \mathbb{R}$. ■

19). $xy' + y = \sqrt{x}, \quad x > 0$

Solution. ■

20).

Solution. This is another first order linear DE. We have

$$y' + \frac{1}{x} y = \frac{1}{\sqrt{x}}$$

We obtain that the integration factor of this is

$$\mu(x) = K' x$$

for some constant $K' \in \mathbb{R}$. So, the solution to our DE is

$$y = \frac{1}{K' x} \int K' x \frac{1}{\sqrt{x}} \, \mathrm{d}x = \frac{1}{x}(\frac{2}{3} x^{3/2} + C)$$

for some $C \in \mathbb{R}$. ■

21). $yy'' - (y')^3 = 0$

Solution. This is an example of a DE with the independent variable missing. We have

$$y'' = \frac{(y')^3}{y}$$

We put $v = y'$, and this gives us the DE

$$v\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{v^3}{y}$$

which is clearly a separable DE. We have

$$\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{1}{y}$$

which, on integrating both sides, gives us

$$\frac{-1}{v} = \ln y + C$$

for some $C \in \mathbb{R}$. So, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v(y) = \frac{-1}{\ln y + C}$$

This one is a *separable* DE. We get

$$-(\ln y + C)\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

and hence this gievs us

$$-(y\ln y - y + Cy) = x + C'$$

where $C, C' \in \mathbb{R}$. ■

22). $2y^2y'' + 2y(y')^2 = 1$

Solution. We assume that $y > 0$ at all points where it is defined. Dividing throughout by $2y^2$, this DE becomes

$$y'' = \frac{1}{2y^2} - \frac{(y')^2}{y}$$

This is a second order equation with the independent variable missing. Putting $v = y'$, we see that

$$v\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{1}{2y^2} - \frac{v^2}{y}$$

which on dividing throughout by v, we rewrite as

$$\frac{\mathrm{d}v}{\mathrm{d}y} + \frac{v}{y} = \frac{1}{2y^2}v^{-1}$$

This is an example of a Bernoulli equation. We substitute $u = v^{1-(-1)} = v^2$ and get

$$\frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{u}{y} = \frac{1}{2y^2}$$

This is a first order linear DE. The integrating factor is

$$\mu(y) = \exp\int\frac{2}{y}\,\mathrm{d}y = \exp\left(2\ln y + K\right) = e^K y^2 = K'y^2$$

where $K' \in \mathbb{R}$. So, the solution to the linear DE is

$$u = \frac{1}{K'y^2}\int K'y^2\frac{1}{y^2}\,\mathrm{d}y = \frac{1}{y^2}\int \mathrm{d}y = \frac{y+C}{y^2}$$

for some $C \in \mathbb{R}$. So, it follows that

$$v^2 = \frac{y+C}{y^2}$$

and hence

$$v = \sqrt{\frac{y+C}{y^2}} = \frac{\sqrt{y+C}}{y}$$

So, we just have to solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{y+C}}{y}$$

which is clearly a separable DE. We get

$$\frac{y}{\sqrt{y+C}}\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

Integrating both sides with respect to x, we get

$$\int \frac{y}{\sqrt{y+C}}\,\mathrm{d}y = \int \mathrm{d}x = x + C_0$$

for some $C_0 \in \mathbb{R}$. The integral on the LHS can be solved by substituting $t = y + C$, and we get

$$\int \frac{y}{\sqrt{y+C}}\,\mathrm{d}y = \int \frac{t-C}{\sqrt{t}}\,\mathrm{d}t = \int \sqrt{t}\,\mathrm{d}t - C\int \frac{1}{\sqrt{t}}\,\mathrm{d}t = \frac{2t^{\frac{3}{2}}}{3} - 2Ct^{\frac{1}{2}} + C'$$

for some $C' \in \mathbb{R}$. So, we get

$$\int \frac{y}{\sqrt{y+C}}\,\mathrm{d}y = \frac{2(y+C)^{\frac{3}{2}}}{3} - 2C(y+C)^{\frac{1}{2}} + C'$$

and combining all this, we see that the solution to our original DE is

$$\frac{2(y+C)^{\frac{3}{2}}}{3} - 2C(y+C)^{\frac{1}{2}} + C' = x + C_0$$

which can be rewritten as

$$\frac{2(y+C)^{\frac{3}{2}}}{3} - 2C(y+C)^{\frac{1}{2}} = x + K$$

where $K = C_0 - C' \in \mathbb{R}$. ■

23). $y' = (1-2x)y^2, \quad y(0) = \frac{-1}{6}$

Solution. This is a separable DE. We have

$$\frac{1}{y^2}y' = (1-2x)$$

Integrating both sides, we get

$$\frac{1}{y^2}\,\mathrm{d}y = \int 1 - 2x\,\mathrm{d}x$$

and this gives us

$$\frac{-1}{y} = x - x^2 + C$$

where $C \in \mathbb{R}$ is some constant. Using the given initial condition, we get $C = 6$. Hence, the solution of the DE is

$$y = \frac{1}{x^2 - x - 6}$$

This is defined on $\mathbb{R}$ minus the roots of the given polynomial. ■

24). $y^2\sqrt{1-x^2}y' = \arcsin x, \ y(0) = 1.$

Solution. This is another example of a *separable* DE. In particular, we have

$$y^2 y' = \frac{\arcsin x}{\sqrt{1-x^2}} \tag{0.3}$$

Now, consider the function $g(y) = y^2$. An antiderivative of $g(y)$ is easily seen to be

$$\int g(y)\mathrm{d}y = \frac{y^3}{3} + K$$

for some $K \in \mathbb{R}$. So, integrating both sides of equation (0.3) with respect to x, we get

$$\int y^2 y' \mathrm{d}x = \int y^2 \mathrm{d}y = \int \frac{\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x$$

Now, the integral on the extreme right hand side in the above equation can be easily calculated by the substitution $t = \arcsin x$, and here we are using the fact that

$$\arcsin' x = \frac{1}{\sqrt{1-x^2}}$$

for $|x| < 1$. So, we get

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x = \int t\mathrm{d}t = \frac{t^2}{2} + K' = \frac{(\arcsin x)^2}{2} + K'$$

for some $K' \in \mathbb{R}$. Combining everything, we see that

$$\int y^2 \mathrm{d}y = \frac{(\arcsin x)^2}{2} + K'$$

and hence we obtain the implicit equation

$$\frac{y^3}{3} = \frac{(\arcsin x)^2}{2} + C$$

for some $C \in \mathbb{R}$. Using the initial condition $y(0) = 1$, we get $C = \frac{1}{3}$. So, the equation is

$$\frac{y^3}{3} = \frac{(\arcsin x)^2}{2} + \frac{1}{3}$$

which can be written as

$$y = \sqrt[3]{\frac{3(\arcsin x)^2}{2} + 1}$$

Now, the cube root function is defined on all of $\mathbb{R}$, and $\arcsin x$ is defined on the interval $[-1, 1]$. Since we are looking for an open interval, the interval of existence in this case is $(-1, 1)$. ■

25). $xye^{x^2y} + x^2e^{x^2y}y' = 0$

Solution. First, let us check whether this DE is exact. We have

$$\frac{\partial(xye^{x^2y})}{\partial y} = xe^{x^2y} + x^3ye^{x^2y}$$

and

$$\frac{\partial(x^2e^{x^2y})}{\partial x} = 2xe^{x^2y} + 2x^3ye^{x^2y}$$

So, this DE is *not* exact. ■

26). $3x^2y\sin(x+y) + x^3y\cos(x+y) + y\sec^2(xy) + (x^3\sin(x+y) + x^3y\cos(x+y) + x\sec^2(xy))y' = 0$

Solution. First, let's check if this DE is exact. We have

$$\frac{\partial(3x^2y\sin(x+y) + x^3y\cos(x+y) + y\sec^2(xy))}{\partial y} = 3x^2\sin(x+y) + 3x^2y\cos(x+y) + x^3\cos(x+y) - x^3y\sin(x+y) + \sec^2(xy) + 2xy\sec^2(xy)\tan(xy)$$

and also

$$\frac{\partial(x^3\sin(x+y) + x^3y\cos(x+y) + x\sec^2(xy))}{\partial x} = 3x^2\sin(x+y) + x^3\cos(x+y) + 3x^2y\cos(x+y) - x^3y\sin(x+y) + \sec^2(xy) + 2xy\sec^2(xy)\tan(xy)$$

and hence we see that

$$\frac{\partial(3x^2y\sin(x+y) + x^3y\cos(x+y) + y\sec^2(xy))}{\partial y} = \frac{\partial(x^3\sin(x+y) + x^3y\cos(x+y) + x\sec^2(xy))}{\partial x}$$

so that the given DE is exact. Now, let

$$P(x,y) = \int x^3\sin(x+y) + x^3y\cos(x+y) + x\sec^2(xy)\,\mathrm{d}x$$

Complete this computation!

■

27). $\alpha ye^{2xy}\mathrm{d}x + (xe^{2xy} + y)\mathrm{d}y = 0.$

Solution. The given DE is equivalent to

$$\alpha ye^{2xy} + (xe^{2xy} + y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

First we have

$$\frac{\partial(\alpha ye^{2xy})}{\partial y} = \alpha e^{2xy} + \alpha 2xye^{2xy} = \alpha e^{2xy}(1+2xy)$$

Next, we have

$$\frac{\partial(xe^{2xy} + y)}{\partial x} = e^{2xy} + 2xye^{2xy} = e^{2xy}(1+2xy)$$

Now for the DE to be exact, the right hand sides of the above two equations must be equal, i.e

$$\alpha e^{2xy}(1+2xy) = e^{2xy}(1+2xy)$$

and hence $\alpha = 1$ makes the given DE exact. ■

28). $(x+y)y^2\mathrm{d}x + (x^2y + \alpha xy^2)\mathrm{d}y = 0.$

Solution. We follow a similar strategy as above. The given DE is equivalent to

$$(x+y)y^2 + (x^2y + \alpha xy^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

First we have

$$\frac{\partial((x+y)y^2)}{\partial y} = 2xy + 3y^2$$

Next, we have

$$\frac{\partial(x^2y + \alpha xy^2)}{\partial x} = 2xy + \alpha y^2$$

Now for the DE to be exact, the right hand sides of the above two equations must be equal, i.e

$$2xy + 3y^2 = 2xy + \alpha y^2$$

and hence $\alpha = 3$ makes the given DE exact. ■