COOKBOOK-2 SOLUTIONS

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1). $2y'' + 3y' - 2y = 0$.

Solution. The characteristic equation for this differential equation is the quadratic

$$2t^2 + 3t - 2 = 0$$

Using the quadratic formula, we can easily obtain that the roots of the quadratic are $-2, 1/2$, and clearly each has multiplicity 1. So, we see that e^{-2x} and $e^{x/2}$ are two linearly independent solutions of this equation. Also, as given in the Cookbook-II, these two elements span the space of solutions of this DE. So, the general solution of this DE is

$$y(x) = c_1 e^{-2x} + c_2 e^{\frac{x}{2}}$$

where $c_1, c_2 \in \mathbb{R}$. ■

2). $9\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 24\frac{\mathrm{d}y}{\mathrm{d}x} + 16y = 0$

Solution. The characteristic polynomial for this equation is

$$9t^2 + 24t + 16 = (3t+4)^2$$

and hence this polynomial has a real root with multiplicty 2, namely $-4/3$. So, the two solutions $e^{\frac{-4}{3}x}$ and $xe^{\frac{-4}{3}x}$ form a basis of the space of solutions to the given DE, and hence the general solution is given by

$$y = c_1 e^{\frac{-4}{3}x} + c_2 x e^{\frac{-4}{3}x}$$

where $c_1, c_2 \in \mathbb{R}$. ■

3). $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 6\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - 10y = 0$, $y(0) = 2$, $y'(0) = -7$ and $y''(0) = 47$

Solution. The characteristic polynomial of the DE is

$$t^3 + 6t^2 + 3t - 10 = (t-1)(t+2)(t+5)$$

and hence this has three distinct roots, each of multiplicity 1. So, the space of solutions has basis e^x, e^{-2x} and e^{-5x}, and hence the general solution is given by

$$y = c_1 e^x + c_2 e^{-2x} + c_3 e^{-5x}$$

We can calculate the constants $c_1, c_2, c_3 \in \mathbb{R}$ using the given initial conditions. The condition $y(0) = 2$ gives us

$$c_1 + c_2 + c_3 = 2$$

The condition $y'(0) = -7$ gives

$$c_1 - 2c_2 - 5c_3 = -7$$

and the condition $y''(0) = 47$ gives us

$$c_1 + 4c_2 + 25c_3 = 47$$

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and solving this system of linear equations in three variables, we get

$$(c_1, c_2, c_3) = (1, -1, 2)$$

and hence

$$y = e^x - e^{-2x} + 2e^{-5x}$$

■

4). $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\dfrac{\mathrm{d}y}{\mathrm{d}x} - 7y = 0$

Solution. The characteristic polynomial of our DE is

$$t^2 - 5t - 7 = \left(t - \frac{5}{2} + \frac{\sqrt{53}}{2}\right)\left(t - \frac{5}{2} - \frac{\sqrt{53}}{2}\right)$$

and hence this polynomial has two real roots, each with multiplicity one. Let these roots be denoted by a, b respectively. So, the general solution is given by

$$y = c_1e^{ax} + c_2e^{bx}$$

with $c_1, c_2 \in \mathbb{R}$. ■

5). $2x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0.$

Solution. This is an example of an *Euler equation*, where we assume $x > 0$. First, note that this equation is equivalent to

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{5}{2}x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0$$

Making the substitution $t = \ln x$, then as mentioned in Cookbook-II we get the DE

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \left(\frac{5}{2} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}t} - y = 0$$

which is the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{3}{2}\frac{\mathrm{d}y}{\mathrm{d}t} - y = 0$$

Now this can be solved as before. The characteristic equation is

$$u^2 + \frac{3}{2}u - 1 = 0$$

It is easily seen that the roots of the polynomial are $-2, 1/2$, and clearly each root has multiplicity 1. So, we see that e^{-2t} and $e^{t/2}$ are two linearly independent solutions of this equation. Also, as given in Cookbook-II, these two elements span the space of solutions of this DE. So, the general solution of this DE is

$$y(t) = c_1e^{-2t} + c_2e^{\frac{t}{2}}$$

where $c_1, c_2 \in \mathbb{R}$. So, to get y in terms of x, we just substitute $t = \ln x$ in the above equation, and we get

$$y(x) = c_1e^{-2\ln x} + c_2e^{\frac{\ln x}{2}} = c_1x^{-2} + c_2x^{\frac{1}{2}}$$

and this is the general solution to this equation. ■

6). $9x^2y'' + 33xy' + 16y = 0$

Solution. This equation is equivalent to

$$x^2y'' + \frac{33}{9}xy' + \frac{16}{9}y = 0$$

Next, we use the substitution $t = \ln x$ to get the equation

$$\frac{d^2y}{dt^2} + \frac{24}{9}\frac{dy}{dt} + \frac{16}{9}y = 0$$

The characteristic equation for this DE is

$$s^2 + \frac{24}{9}s + \frac{16}{9} = 0$$

which is the same as the equation

$$9s^2 + 24s + 16 = 0$$

and the only root is $-4/3$ with multiplicity 2. So, the general solution is given by

$$y(t) = c_1e^{\frac{-4t}{3}} + c_2te^{\frac{-4}{3}t}$$

with $c_1, c_2 \in \mathbb{R}$, and hence in terms of x, we have

$$y = c_1x^{\frac{-4}{3}} + c_2x^{\frac{-4}{3}}\ln x$$

■

7). $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 7y = 0$

Solution. Using the substitution $t = \ln x$ we get

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} - 7y = 0$$

The characteristic equation of this DE is

$$s^2 - 5s - 7 = 0$$

which is the same as the one in problem **4).** From there, we can get the solution $y(t)$, which can be easily converted to $y(x)$. ■

8). $x\frac{d^2y}{dx^2} + (x^2 - 1)\frac{dy}{dx} + x^3y = 0, \quad x > 0$

Solution. Dividing throughout by x, we get the equation

$$\frac{dy^2}{dx^2} + \frac{(x^2-1)}{x}\frac{dy}{dx} + x^2y = 0$$

Put

$$p(x) = \frac{(x^2-1)}{x} \quad , \quad q(x) = x^2$$

Observe that

$$\frac{q'(x) + 2p(x)q(x)}{2(q(x))^{3/2}} = \frac{2x + 2x(x^2-1)}{2x^3} = 1$$

i.e this quantity is constant on the interval of existence. So, consider the transformation

$$t = \int \sqrt{q(x)}\,dx = \int x\,dx = \frac{x^2}{2}$$

For these transformations we have

$$(*) \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}$$

n and

$$(**) \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}x^2}$$

In our case, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{2t}\frac{\mathrm{d}y}{\mathrm{d}t} \quad , \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t}$$

and so the equation becomes

$$2t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2t-1}{\sqrt{2t}}\sqrt{2t}\frac{\mathrm{d}y}{\mathrm{d}t} + 2ty = 0$$

which is the same as

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + y = 0$$

Note this this is a DE with constant coefficients, and this can be easily solved. The characteristic equation is

$$s^2 + s + 1 = 0$$

The roots of this polynomial are $\frac{-1\pm i\sqrt{3}}{2}$, and each root has multiplicity 1. So the general solution of the DE is given by

$$y(t) = c_1e^{\frac{-t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_2e^{\frac{-t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

where $c_1, c_2 \in \mathbb{R}$ are constants. So, in terms of x, the general solution is

$$y = c_1e^{\frac{-x^2}{4}}\cos\left(\frac{\sqrt{3}}{4}x^2\right) + c_2e^{\frac{-x^2}{4}}\sin\left(\frac{\sqrt{3}}{4}x^2\right)$$

where $c_1, c_2 \in \mathbb{R}$. ■

9). $y^{(6)} - 3y^{(5)} + 40y^{(3)} - 180y'' + 324y' - 432y = 0$ [Hint: $1 + i\sqrt{5}$ is a root (with positive multiplicity) of the characteristic polynomial.]

Solution. The characteristic polynomial of the DE is

$$t^6 - 3t^5 + 40t^3 - 180t^2 + 324t - 432$$

As given in the hint, $1 + i\sqrt{5}$ is a root, and because it occurs as a conjugate pair we see that $(t - 1 - i\sqrt{5})(t - 1 + i\sqrt{5}) = (t-1)^2 + 5$ is a factor of the given polynomial. Using long division, we get

$$t^6 - 3t^5 + 40t^3 - 180t^2 + 324t - 432 = ((t-1)^2 + 5)(t^4 - t^3 - 8t^2 + 30t - 72)$$

Now, it can be checked that $3, -4$ are roots of the biquadratic factor, i.e $(t-3)(t+4) = t^2 + t - 12$ is a factor of the above biquadratic factor. Again by long division we see that

$$t^4 - t^3 - 8t^2 + 30t - 72 = (t^2 + t - 12)(t^2 - 2t + 6)$$

Finally, the polynomial $t^2 - 2t + 6$ has $1 + i\sqrt{5}, 1 - i\sqrt{5}$ as its roots. So, it follows that our original polynomial $t^6 - 3t^5 + 40t^3 - 180t^2 + 324t - 432$ has the following roots:

$$\begin{aligned} 1 + i\sqrt{5} &\text{ with multiplicity } 2 \\ 1 - i\sqrt{5} &\text{ with multiplicity } 2 \\ 3 &\text{ with multiplicity } 1 \\ -4 &\text{ with multiplicity } 1 \end{aligned}$$

So, it follows that the basis of the space of solutions consists of

$$e^{3x}, e^{-4x}, e^x \cos(\sqrt{5}x), xe^x \cos(\sqrt{5}x), e^x \sin(\sqrt{5}x), xe^x \sin(\sqrt{5}x)$$

So, the general solution to this DE is given by

$$y = c_1 e^{3x} + c_2 e^{-4x} + c_3 e^x \cos(\sqrt{5}x) + c_4 xe^x \cos(\sqrt{5}x) + c_5 e^x \sin(\sqrt{5}x) + c_6 xe^x \sin(\sqrt{5}x)$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{R}$ are constants. ■

10). $4y'' + 11y' + 6y = 0, \quad y(0) = 5, \quad y'(0) = -5$

Solution. The characteristic equation of the polynomial is

$$4t^2 + 11t + 6 = 0$$

The roots are -2 and $-3/4$ with multiplicity 1 each. So, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{\frac{-3x}{4}}$$

where $c_1, c_2 \in \mathbb{R}$. Using the two initial conditions, the values of the constants c_1, c_2 can easily be obtained. ■

11). $3y'' - 2y' - 2y = \cos(2x)$

Solution. We can write

$$3y'' - 2y' - 2y = e^{0x}(1 \cdot \cos(2x) + 0 \cdot \sin(2x))$$

so that $\alpha = 0, \beta = 2$ where α, β are as in Cookbook-II. Now, the characteristic equation is

$$3t^2 - 2t - 2 = 0$$

and it is clear that $r = 0 + 2i$ is *not* a root of the characteristic polynomial. So a particular solution of the above DE is of the form

$$y_p = A(x)\cos(2x) + B(x)\sin(2x)$$

where $A(x), B(x)$ are polynomials of degree 0, i.e they are constants. So we have

$$y_p = A\cos(2x) + B\sin(2x)$$

Using the DE, we can figure out the constants. We get

$$(-14A - 4B)\cos(2x) + (4A - 14B)\sin(2x) = \cos(2x)$$

and hence A, B can be obtained. ■

12). $2y'' + 7y' + 6 = e^{-2x}$

Solution. We can write this DE as

$$2y'' + 7y' + 6 = e^{-2x}(1 \cdot \cos(0x) + 0 \cdot \sin(0x))$$

and hence $\alpha = -2, \beta = 0$ where α, β are as in Cookbook-II. Now, the multiplicity of -2 as a root of the characteristic polynomial

$$2t^2 + 7t + 6$$

is 1. So, the form of a particular solution is

$$y_p = xe^{-2x}(A(x)\cos(0x) + B(x)\sin(0x)) = xe^{-2x}A(x)$$

where $A(x)$ is a polynomial of degree 0, i.e it is a constant. So,

$$y_p = Axe^{-2x}$$

for some $A \in \mathbb{R}$. The constant A can be figured out from the DE. ■

13). $2x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = x^2 - 3\sqrt{x}$

Solution. First divide throughout by 2 to get the equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{5}{2}x\frac{\mathrm{d}y}{\mathrm{d}x} - y = \frac{x^2 - 3\sqrt{x}}{2}$$

Let us use the substitution $t = \ln x$ to get the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{3}{2}\frac{\mathrm{d}y}{\mathrm{d}t} - y = \frac{e^{2t} - 3e^{\frac{t}{2}}}{2}$$

We can split this equation into two equations and use the *principle of superposition*. The equations are as follows.

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{3}{2}\frac{\mathrm{d}y}{\mathrm{d}t} - y = \frac{e^{2t}}{2} \tag{0.1}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{3}{2}\frac{\mathrm{d}y}{\mathrm{d}t} - y = \frac{-3e^{\frac{t}{2}}}{2} \tag{0.2}$$

We can find particular solutions to these two equations separately, and then just add them to obtain a particular solution of our original DE. Might complete this later. ■

14). $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \sin\,(\ln\,x) + x^2$

Solution. We assume that $x > 0$. First, by using the substitution $t = \ln\,x$, we convert this equation to a more convenient form (as given in Cookbook-II):

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = \sin\,t + e^{2t} \tag{0.3}$$

Now, we will use the *principle of superposition* to find the general solution of this DE. For this, consider the following three DEs:

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 0 \tag{0.4}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = \sin\, t \tag{0.5}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = e^{2t} \tag{0.6}$$

Equation (0.4) is the homogeneous version equation (0.3). If we are able to find solutions to (0.5) and (0.6), then by adding them we can get a particular solution of equation (0.3), and here is where we are applying the *principle of superposition*. Then, if we are able to find a general solution to equation (0.4), then the general solution of (0.3) will be the sum of the particular solution found and the general form of the solution of (0.4) (all of this is given in Cookbook-II). So this will be our strategy.

The characteristic polynomial for the DE in (0.4) is

$$u^2 - 3u + 2 = (u-1)(u-2)$$

and it has two roots, namely $1, 2$ each with multiplicity 1. So, the general solution to (0.4) is

$$y = c_1e^t + c_2e^{2t}$$

for $c_1, c_2 \in \mathbb{R}$.

Next, we focus on equation (0.5). Observe that

$$\sin\, t = e^{0t}\left(0 \cdot \cos(1 \cdot t) + 1 \cdot \sin(1 \cdot t)\right)$$

and hence as in Cookbook-II, we have $\alpha = 0$ and $\beta = 1$ in this case. Clearly, $\alpha + i\beta = i$ is *not* a root of the characteristic polynomial, so we are in the non-resonance case. So, a particular solution of (0.5) is of the form

$$y = e^{0 \cdot t}\left(A(t)\cos(1 \cdot t) + B(t)\sin(1 \cdot t)\right)$$

where $A(t), B(t)$ are polynomials of degree 0, i.e they are constant polynomials. So a particular solution of (0.5) is of the form

$$y = k_1\cos\, t + k_2\sin\, t$$

where $k_1, k_2 \in \mathbb{R}$. We now compute k_1, k_2 from equation (0.5). We get

$$(-k_1\cos\, t - k_2\sin\, t) - 3(-k_1\sin\, t + k_2\cos\, t) + 2(k_1\cos\, t + k_2\sin\, t) = \sin\, t$$

and this implies that

$$(k_1 - 3k_2)\cos\, t + (k_2 + 3k_1)\sin\, t = \sin\, t$$

and we obtain $k_2 = 1/10$ and $k_1 = 3/10$. So, a particular solution of (0.5) is

$$y = \frac{3\cos\, t}{10} + \frac{\sin\, t}{10}$$

Finally, we focus on (0.6). Observe that

$$e^{2t} = e^{2t}\left(1 \cdot \cos(0 \cdot t) + 1 \cdot \sin(0 \cdot t)\right)$$

and hence in this case $\alpha = 2$ and $\beta = 0$. Now, note that $\alpha + i\beta = 2$ is a root of the characteristic polynomial and it is multiplicity 1. So, a particular solution of (0.6) is of the form

$$y = te^{2t}(A(t)\cos(0 \cdot t) + B(t)\sin(0 \cdot t))$$

where $A(t), B(t)$ are polynomials of degree 0, i.e they are constant polynomials. So a particular solution of (0.6) is of the form

$$y = kte^{2t}$$

where $k \in \mathbb{R}$ is some constant. We can compute k from equation (0.6).

$$4ke^{2t} + 4kte^{2t} - 3[ke^{2t} + 2kte^{2t}] + 2kte^{2t} = e^{2t}$$

and this equation implies

$$ke^{2t} = e^{2t}$$

and we get $k = 1$. So, a particular solution of (0.6) is

$$y = te^{2t}$$

Combining solutions to (0.4), (0.5) and (0.6) we see that the general solution to (0.3) is

$$y = c_1e^t + c_2e^{2t} + \frac{3\cos t}{10} + \frac{\sin t}{10} + te^{2t}$$

for some $c_1, c_2 \in \mathbb{R}$. Finally, using our original substitution $t = \ln x$, we see that the general solution to our DE is

$$y = c_1x + c_2x^2 + \frac{3\cos(\ln x)}{10} + \frac{\sin(\ln x)}{10} + x^2\ln x$$

■

15). $y'' - 2y' + 17 = (3x^2 + 2)e^x \sin(2x)$

Solution. To be completed. ■

16). $x\frac{d^2y}{dx^2} + (x^2 - 1)\frac{dy}{dx} + x^3y = \exp(-\frac{1}{4}x^2)\cos(\frac{\sqrt{3}}{4}x^2), \quad x > 0$

Solution. To be completed. ■

17). $y^{(4)} - 4y^{(3)} + 38y'' - 68y + 289 = xe^x \cos(2x)$

Solution. To be completed. ■

18). $y^{(6)} - 3y^{(5)} + 45y^{(4)} - 24y^{(3)} + 236y'' + 1300y' - 4056y = x^2e^x \cos(\sqrt{5}x)$

Solution. First, we write the given DE in the form

$$y^{(6)} - 3y^{(5)} + 45y^{(4)} - 24y^{(3)} + 236y'' + 1300y' - 4056y = e^{1\cdot x}(x^2 \cdot \cos(\sqrt{5}x) + 0 \cdot \sin(\sqrt{5}x))$$

and hence $\alpha = 1, \beta = \sqrt{5}$ where α, β are as in Cookbook-II. So, the *form* of a particular solution of this DE will be

$$y_p = x^se^x \left(A(x)\cos(\sqrt{5}x) + B(x)\sin(\sqrt{5}x)\right)$$

where A, B are polynomials of degree 2 and s is the multiplicity of $r = 1+i\sqrt{5}$ as a root of the characteristic polynomial. Let us calculate this multiplicity. The characteristic polynomial is

$$t^6 - 3t^5 + 45t^4 - 24t^3 + 236t^2 + 1300t - 4056$$

If $1 + i\sqrt{5}$ is a root, then $1 - i\sqrt{5}$ is also a root, and hence $(x - 1)^2 + 5 = x^2 - 2x + 6$ must be a factor of this polynomial. But this is not true, and hence the multiplicity $s = 0$. ■

19). We will show that the substitution $t = \ln x$ transforms the equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \alpha x \frac{\mathrm{d}y}{\mathrm{d}x} + \beta y = 0, \quad x > 0$$

to the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (\alpha - 1)\frac{\mathrm{d}y}{\mathrm{d}t} + \beta y = 0$$

Solution. We will use equations $(*)$ and $(**)$, and the rest is just a straightforward computation. We see that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{1}{x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{1}{e^t}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\frac{1}{e^{2t}} - \frac{\mathrm{d}y}{\mathrm{d}t}\frac{1}{e^{2t}}$$

So, our equation becomes

$$e^{2t}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\frac{1}{e^{2t}} - \frac{\mathrm{d}y}{\mathrm{d}t}\frac{1}{e^{2t}}\right) + \alpha e^t \frac{\mathrm{d}y}{\mathrm{d}t}\frac{1}{e^t} + \beta y = 0$$

and this is the same as the desired equation. ∎

20).

Solution. To be completed. ∎