# HW-1

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**1).** Here we will be solving problems **3** and **24** from Cookbook-I.

**3** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3x^2 - 2x - 1}{2(y-1)}, \ y(3) = 1 - \sqrt{13}.$

**Solution.** This is easily seen to be a *separable* DE. In particular, we have

$$(2y-2)\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2x - 1 \tag{0.1}$$

Now, consider the function $g(y) = 2y - 2$. The antiderivative of $g(y)$ is

$$\int g(y)\mathrm{d}y = y^2 - 2y + K$$

for some $K \in \mathbb{R}$. So, integrating both sides of equation (0.1) with respect to $x$, we get

$$\int (2y-2)\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int (2y-2)\mathrm{d}y = \int (3x^2 - 2x - 1)\mathrm{d}x$$

and hence we obtain

$$y^2 - 2y = x^3 - x^2 - x + C$$

for some $C \in \mathbb{R}$. Using the initial condition $y(3) = 1 - \sqrt{13}$, we get

$$(1-\sqrt{13})^2 - 2(1-\sqrt{13}) = 27 - 9 - 3 + C$$

and from here we obtain $C = -3$. Hence, we obtain the implicit equation

$$y^2 - 2y - x^3 + x^2 + x + 3 = 0$$

■

**24** $y^2\sqrt{1-x^2}y' = \arcsin\, x, \ y(0) = 1.$

**Solution.** This is another example of a *separable* DE. In particular, we have

$$y^2 y' = \frac{\arcsin\, x}{\sqrt{1-x^2}} \tag{0.2}$$

Now, consider the function $g(y) = y^2$. An antiderivative of $g(y)$ is easily seen to be

$$\int g(y)\mathrm{d}y = \frac{y^3}{3} + K$$

for some $K \in \mathbb{R}$. So, integrating both sides of equation (0.2) with respect to $x$, we get

$$\int y^2 y' \mathrm{d}x = \int y^2 \mathrm{d}y = \int \frac{\arcsin\, x}{\sqrt{1-x^2}}\mathrm{d}x$$

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Now, the integral on the extreme right hand side in the above equation can be easily calculated by the substitution $t = \arcsin x$, and here we are using the fact that

$$\arcsin' x = \frac{1}{\sqrt{1-x^2}}$$

for $|x| < 1$. So, we get

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x = \int t\mathrm{d}t = \frac{t^2}{2} + K' = \frac{(\arcsin x)^2}{2} + K'$$

for some $K' \in \mathbb{R}$. Combining everything, we see that

$$\int y^2 \mathrm{d}y = \frac{(\arcsin x)^2}{2} + K'$$

and hence we obtain the implicit equation

$$\frac{y^3}{3} = \frac{(\arcsin x)^2}{2} + C$$

for some $C \in \mathbb{R}$. Using the initial condition $y(0) = 1$, we get $C = \frac{1}{3}$. So, the equation is

$$\frac{y^3}{3} = \frac{(\arcsin x)^2}{2} + \frac{1}{3}$$

which can be written as

$$y = \sqrt[3]{\frac{3(\arcsin x)^2}{2} + 1}$$

Now, the cube root function is defined on all of $\mathbb{R}$, and $\arcsin x$ is defined on the interval $[-1, 1]$. Since we are looking for an open interval, the interval of existence in this case is $(-1, 1)$. ■

**2).** Here I will be solving problems **27** and **28** from Cookbook-I.

**27** $\alpha y e^{2xy}\mathrm{d}x + (xe^{2xy} + y)\mathrm{d}y = 0.$

**Solution**. The given DE is equivalent to

$$\alpha y e^{2xy} + (xe^{2xy} + y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

First we have

$$\frac{\partial(\alpha y e^{2xy})}{\partial y} = \alpha e^{2xy} + \alpha 2xye^{2xy} = \alpha e^{2xy}(1 + 2xy)$$

Next, we have

$$\frac{\partial(xe^{2xy} + y)}{\partial x} = e^{2xy} + 2xye^{2xy} = e^{2xy}(1 + 2xy)$$

Now for the DE to be exact, the right hand sides of the above two equations must be equal, i.e

$$\alpha e^{2xy}(1 + 2xy) = e^{2xy}(1 + 2xy)$$

and hence $\alpha = 1$ makes the given DE exact. ■

**28** $(x + y)y^2\mathrm{d}x + (x^2y + \alpha xy^2)\mathrm{d}y = 0.$

**Solution**. We follow a similar strategy as above. The given DE is equivalent to

$$(x+y)y^2 + (x^2y + \alpha xy^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

First we have

$$\frac{\partial((x+y)y^2)}{\partial y} = 2xy + 3y^2$$

Next, we have

$$\frac{\partial(x^2y + \alpha xy^2)}{\partial x} = 2xy + \alpha y^2$$

Now for the DE to be exact, the right hand sides of the above two equations must be equal, i.e

$$2xy + 3y^2 = 2xy + \alpha y^2$$

and hence $\alpha = 3$ makes the given DE exact. ■

**3).** Here I will be solving problems **1** and **5** from Cookbook-II.

**1**. $2y'' + 3y' - 2y = 0$.

**Solution**. The characteristic equation for this differential equation is the quadratic

$$2t^2 + 3t - 2 = 0$$

Using the quadratic formula, we can easily obtain that the roots of the quadratic are $-2, 1/2$, and clearly each has multiplicity 1. So, we see that $e^{-2x}$ and $e^{x/2}$ are two linearly independent solutions of this equation. Also, as given in the Cookbook-II, these two elements span the space of solutions of this DE. So, the general solution of this DE is

$$y(x) = c_1e^{-2x} + c_2e^{\frac{x}{2}}$$

where $c_1, c_2 \in \mathbb{R}$. ■

**5**. $2x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0$.

**Solution**. This is an example of an *Euler equation*, where we assume $x > 0$. First, note that this equation is equivalent to

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{5}{2}x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0$$

Making the substitution $t = \ln x$, then as mentioned in Cookbook-II we get the DE

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \left(\frac{5}{2} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}t} - y = 0$$

which is the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{3}{2}\frac{\mathrm{d}y}{\mathrm{d}t} - y = 0$$

Now this can be solved as before. The characteristic equation is

$$u^2 + \frac{3}{2}u - 1 = 0$$

It is easily seen that the roots of the polynomial are $-2, 1/2$, and clearly each root has multiplicity 1. So, we see that $e^{-2t}$ and $e^{t/2}$ are two linearly independent solutions of this equation. Also, as given in Cookbook-II, these two elements span the space of solutions of this DE. So, the general solution of this DE is

$$y(t) = c_1 e^{-2t} + c_2 e^{\frac{t}{2}}$$

where $c_1, c_2 \in \mathbb{R}$. So, to get $y$ in terms of $x$, we just substitute $t = \ln x$ in the above equation, and we get

$$y(x) = c_1 e^{-2\ln x} + c_2 e^{\frac{\ln x}{2}} = c_1 x^{-2} + c_2 x^{\frac{1}{2}}$$

and this is the general solution to this equation. ■

Before solving problems **4)** and **5)**, let us first write down all the DEs.

$$(*) \qquad \dot{\vec{\boldsymbol{x}}} = \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) \quad , \quad \vec{\boldsymbol{x}}(t_0) = \vec{\boldsymbol{x}}_0$$

$$(*_{\text{tr}}) \qquad \dot{\vec{\boldsymbol{x}}} = -\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) \quad , \quad \vec{\boldsymbol{x}}(t_0) = \vec{\boldsymbol{x}}_0$$

$$(*_{\text{sr}}) \qquad \dot{\vec{\boldsymbol{x}}} = \vec{\boldsymbol{v}}^{\text{sr}}(\vec{\boldsymbol{x}}) \quad , \quad \vec{\boldsymbol{x}}(t_0) = -\vec{\boldsymbol{x}}_0$$

where the map $\vec{\boldsymbol{v}} : \boldsymbol{\Omega} \to \mathbb{R}^n$ is $\mathscr{C}^1$ and the map $\vec{\boldsymbol{v}}^{\text{sr}} : -\boldsymbol{\Omega} \to \mathbb{R}^n$ is given by

$$\vec{\boldsymbol{v}}^{\text{sr}}(\vec{\boldsymbol{x}}) = -\vec{\boldsymbol{v}}(-\vec{\boldsymbol{x}})$$

**4).** (For this problem, we drop the bold and arrow notation since we are living in dimension 1, but the proof remains valid in higher dimensions as well) Let $\varphi : (a, b) \to \mathbb{R}$ be a solution of $(*)$, and let $\varphi^{\text{tr}}$, $\varphi^{\text{sr}}$ be the state and time reversals of $\varphi$ respectively. We show that $\varphi^{\text{tr}}$ and $\varphi^{\text{sr}}$ are solutions of $(*_{\text{tr}})$ and $(*_{\text{sr}})$ respectively.

First, note that $\varphi^{\text{tr}} : (2t_0 - b, 2t_0 - a) \to \boldsymbol{\Omega}$ is defined by

$$\varphi^{\text{tr}}(t) = \varphi(2t_0 - t) \quad , \quad 2t_0 - b < t < 2t_0 - a$$

Now, observe that

$$\varphi^{\text{tr}}(t_0) = \varphi(2t_0 - t_0) = \varphi(t_0) = x_0$$

and so the boundary condition is satisfied. By the chain rule, we see that

$$(\varphi^{\text{tr}})'(t) = -\varphi'(2t_0 - t) = -v(\varphi(2t_0 - t)) = -v(\varphi^{\text{tr}}(t))$$

and written in dot notation, this implies that

$$\dot{\varphi}^{\text{tr}}(t) = -v(\varphi^{\text{tr}}(t)) \quad , \quad 2t_0 - b < t < 2t_0 - a$$

and combined with the boundary condition, we conclude that $\varphi^{\text{tr}}$ satisfies the equation $(*_{\text{tr}})$.

Next, let us focus on $\varphi^{\text{sr}}$, the state reversal of $\varphi$. Note that $\varphi^{\text{sr}} : (a, b) \to -\boldsymbol{\Omega}$ is defined by

$$\varphi^{\text{sr}}(t) = -\varphi(t) \quad , \quad t \in (a, b)$$

This immediately gives us the boundary condition

$$\varphi^{\text{sr}}(t_0) = -\varphi(t_0) = -x_0$$

By simple differentiation, for any $t \in (a, b)$ we have

$$(\varphi^{\text{sr}})'(t) = -\varphi'(t) = -v(\varphi(t)) = v^{\text{sr}}(-\varphi(t)) = v^{\text{sr}}(\varphi^{sr}(t))$$

Using the dot notation, this equation can be written as

$$\dot{\varphi}^{\text{sr}}(t) = v^{\text{sr}}(\varphi^{\text{sr}}(t)) \quad , \quad t \in (a, b)$$

and combining this with the boundary condition, we conclude that $\varphi^{\text{sr}}$ satisfies the equation $(*_{\text{sr}})$. This completes the solution to our problem.

**5).** We show the given identities.

**(a)**. $(\vec{\varphi}^{\text{tr}})^{\text{tr}} = \vec{\varphi}$. Since $\vec{\varphi}^{\text{tr}}$ is defined on the interval $(2t_0 - b, 2t_0 - a)$, we see that $(\vec{\varphi}^{\text{tr}})^{\text{tr}}$ is defined on the interval

$$(2t_0 - 2t_0 + a, 2t_0 - 2t_0 + b) = (a, b)$$

So, let $t \in (a, b)$. We have

$$(\vec{\varphi}^{\text{tr}})^{\text{tr}}(t) = \vec{\varphi}^{\text{tr}}(2t_0 - t) = \vec{\varphi}(2t_0 - (2t_0 - t)) = \vec{\varphi}(t)$$

and this proves the given equality.

**(b)**. $(\vec{\varphi}^{\text{sr}})^{\text{sr}} = \vec{\varphi}$. Because $\vec{\varphi}^{\text{sr}}$ is defined on the interval $(a, b)$, it follows that $(\vec{\varphi}^{\text{sr}})^{\text{sr}}$ is also defined on $(a, b)$. So, let $t \in (a, b)$. By the definition of state reversal, we see that

$$(\vec{\varphi}^{\text{sr}})^{\text{sr}}(t) = -\vec{\varphi}^{\text{sr}}(t) = -(-\vec{\varphi}(t)) = \vec{\varphi}(t)$$

and hence this proves the desired equality.

**(c)**. $(\vec{\varphi}^{\text{sr}})^{\text{tr}} = (\vec{\varphi}^{\text{tr}})^{\text{sr}}$. First, observe that $\vec{\varphi}^{\text{sr}}$ is defined on the interval $(a, b)$, and hence it follows that $(\vec{\varphi}^{\text{sr}})^{\text{tr}}$ is defined on the interval $(2t_0 - b, 2t_0 - a)$. Similarly, $\vec{\varphi}^{\text{tr}}$ is defined on the interval $(2t_0 - b, 2t_0 - a)$, and hence $(\vec{\varphi}^{\text{tr}})^{\text{sr}}$ is defined on the interval $(2t_0 - b, 2t_0 - a)$. So, we've shown that the domains of the functions match. Now, let $t \in (2t_0 - b, 2t_0 - a)$. So, we have the following chain of equalities.

$$(\vec{\varphi}^{\text{sr}})^{\text{tr}}(t) = \vec{\varphi}^{\text{sr}}(2t_0 - t) = -\vec{\varphi}(2t_0 - t) = -\vec{\varphi}^{\text{tr}}(t) = (\vec{\varphi}^{\text{tr}})^{\text{sr}}(t)$$

and this proves the desired equality. So, this completes the solution to the problem.