HW-2

SIDDHANT CHAUDHARY

1). Here I will solve problems **1** and **12** from Cookbook-I.

1 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1+y^2}{x}, \quad x > 0$

Solution. This is seen to be a separable DE. We have

$$\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

So, integrating both sides with respect to x, we see that

$$\int \frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int \frac{1}{1+y^2}\mathrm{d}y = \int \frac{1}{x}\mathrm{d}x = \ln\, x + K$$

for some $K \in \mathbb{R}$. Now,

$$\int \frac{1}{1+y^2}\mathrm{d}y = \arctan\, y + K'$$

for some $K' \in \mathbb{R}$. Combining all of this, we get

$$\arctan\, y = \ln\, x + C$$

for some $C \in \mathbb{R}$, and this is the general solution of the DE. ■

12 $\sin\, x\dfrac{\mathrm{d}y}{\mathrm{d}x} + (\cos\, x)y = e^x$

Solution. For simplicity we assume that $x \in (0, \pi)$. So, dividing throughout by the sine term, this becomes a linear first order DE.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + (\cot\, x)y = \frac{e^x}{\sin\, x}$$

So, the integrating factor $\mu(x)$ in our case is

$$\mu(x) = \exp \int \cot\, x\mathrm{d}x = \exp\, [\ln\, \sin\, x + K]$$

for some $K \in \mathbb{R}$. So, the integrating factor is

$$\mu(x) = e^K e^{\ln\, \sin\, x} = K' e^{\ln\, \sin\, x} = K'\sin\, x$$

So, the general solution to this DE is given by

$$y = \mu(x)^{-1} \int \mu(x)\frac{e^x}{\sin\, x}\mathrm{d}x = \frac{1}{K'\sin\, x}\int K'\sin\, x\frac{e^x}{\sin\, x}\mathrm{d}x = \frac{e^x + C}{\sin\, x}$$

for some $C \in \mathbb{R}$. ■

Date: January 2021.

2). Here I will be solving problems **2** and **14** from Cookbook-II.

2 $9\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 24\frac{\mathrm{d}y}{\mathrm{d}x} + 16y = 0$

Solution. The characteristic polynomial for this equation is

$$9t^2 + 24t + 16 = (3t+4)^2$$

and hence this polynomial has a real root with multiplicty 2, namely $-4/3$. So, the two solutions $e^{\frac{-4}{3}x}$ and $xe^{\frac{-4}{3}x}$ form a basis of the space of solutions to the given DE, and hence the general solution is given by

$$y = c_1e^{\frac{-4}{3}x} + c_2xe^{\frac{-4}{3}x}$$

where $c_1, c_2 \in \mathbb{R}$. ∎

14 $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \sin(\ln x) + x^2$

Solution. We assume that $x > 0$. First, by using the substitution $t = \ln x$, we convert this equation to a more convenient form (as given in Cookbook-II):

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = \sin t + e^{2t} \tag{0.1}$$

Now, we will use the *principle of superposition* to find the general solution of this DE. For this, consider the following three DEs:

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 0 \tag{0.2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = \sin t \tag{0.3}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = e^{2t} \tag{0.4}$$

Equation (0.2) is the homogeneous version equation (0.1). If we are able to find solutions to (0.3) and (0.4), then by adding them we can get a particular solution of equation (0.1), and here is where we are applying the *principle of superposition*. Then, if we are able to find a general solution to equation (0.2), then the general solution of (0.1) will be the sum of the particular solution found and the general form of the solution of (0.2) (all of this is given in Cookbook-II). So this will be our strategy.

The characteristic polynomial for the DE in (0.2) is

$$u^2 - 3u + 2 = (u-1)(u-2)$$

and it has two roots, namely $1, 2$ each with multiplicity 1. So, the general solution to (0.2) is

$$y = c_1e^t + c_2e^{2t}$$

for $c_1, c_2 \in \mathbb{R}$.

Next, we focus on equation (0.3). Observe that

$$\sin t = e^{0t}\left(0\cdot\cos(1\cdot t) + 1\cdot\sin(1\cdot t)\right)$$

and hence as in Cookbook-II, we have $\alpha = 0$ and $\beta = 1$ in this case. Clearly, $\alpha + i\beta = i$ is *not* a root of the characteristic polynomial, so we are in the non-resonance case. So, a particular solution of (0.3) is of the form

$$y = e^{0\cdot t}\left(A(t)\cos(1\cdot t) + B(t)\sin(1\cdot t)\right)$$

where $A(t), B(t)$ are polynomials of degree 0, i.e they are constant polynomials. So a particular solution of (0.3) is of the form

$$y = k_1\cos\, t + k_2\sin\, t$$

where $k_1, k_2 \in \mathbb{R}$. We now compute k_1, k_2 from equation (0.3). We get

$$(-k_1\cos\, t - k_2\sin\, t) - 3(-k_1\sin\, t + k_2\cos\, t) + 2(k_1\cos\, t + k_2\sin\, t) = \sin\, t$$

and this implies that

$$(k_1 - 3k_2)\cos\, t + (k_2 + 3k_1)\sin\, t = \sin\, t$$

and we obtain $k_2 = 1/10$ and $k_1 = 3/10$. So, a particular solution of (0.3) is

$$y = \frac{3\cos\, t}{10} + \frac{\sin\, t}{10}$$

Finally, we focus on (0.4). Observe that

$$e^{2t} = e^{2t}\left(1\cdot\cos(0\cdot t) + 1\cdot\sin(0\cdot t)\right)$$

and hence in this case $\alpha = 2$ and $\beta = 0$. Now, note that $\alpha + i\beta = 2$ is a root of the characteristic polynomial and it is multiplicity 1. So, a particular solution of (0.4) is of the form

$$y = te^{2t}(A(t)\cos(0\cdot t) + B(t)\sin(0\cdot t))$$

where $A(t), B(t)$ are polynomials of degree 0, i.e they are constant polynomials. So a particular solution of (0.4) is of the form

$$y = kte^{2t}$$

where $k \in \mathbb{R}$ is some constant. We can compute k from equation (0.4).

$$4ke^{2t} + 4kte^{2t} - 3[ke^{2t} + 2kte^{2t}] + 2kte^{2t} = e^{2t}$$

and this equation implies

$$ke^{2t} = e^{2t}$$

and we get $k = 1$. So, a particular solution of (0.4) is

$$y = te^{2t}$$

Combining solutions to (0.2), (0.3) and (0.4) we see that the general solution to (0.1) is

$$y = c_1e^t + c_2e^{2t} + \frac{3\cos\, t}{10} + \frac{\sin\, t}{10} + te^{2t}$$

for some $c_1, c_2 \in \mathbb{R}$. Finally, using our original substitution $t = \ln\, x$, we see that the general solution to our DE is

$$y = c_1x + c_2x^2 + \frac{3\cos(\ln\, x)}{10} + \frac{\sin(\ln\, x)}{10} + x^2\ln\, x$$

$\blacksquare$

3). The equation we have is

$$t^2y'' + 4ty' - 10y = 0$$

Suppose $y = t^r$ is a solution to this DE. Then, we have

$$t^2(r(r-1)t^{r-2}) + 4t(rt^{r-1}) - 10t^r = 0$$

which implies that

$$t^r[r(r-1) + 4r - 10] = 0$$

and hence

$$r^2 + 3r - 10 = (r+5)(r-2) = 0$$

and hence the values of r which satisfy the equation are $r = -5, 2$.

4). Consider the DE

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + ... + p_n(x)y(x) = 0$$

in the interval (a, b), where each p_i is continuous on (a, b). We show that the space of solutions of this DE is an $\mathbb{R}$-vector space of dimension n. We will be using **Theorem 1** as mentioned in the homework sheet.

Let $t_0 \in (a, b)$ be the midpoint of the interval (a, b), and let S be the space of solutions of the given DE. We will produce a vector space isomorphism $\varphi : S \to \mathbb{R}^n$ which is defined as follows: for any $y \in S$, we define the map φ as

$$\varphi(y) = (y(t_0), y^{(1)}(t_0), y^{(2)}(t_0), ..., y^{(n-1)}(t_0))$$

First, we show that this map preserves addition and scalar multiplication. So, let $y_1, y_2 \in S$ and let $c \in \mathbb{R}$. It is clear that $y_1 + y_2$ and cy_1 are solutions of the given DE, because the derivative as an operator is a linear operator. Also, observe that

$$\begin{aligned}
\varphi(y_1 + y_2) &= ((y_1 + y_2)(t_0), (y_1 + y_2)^{(1)}(t_0), ..., (y_1 + y_2)^{(n-1)}(t_0)) \\
&= (y_1(t_0) + y_2(t_0), y_1^{(1)}(t_0) + y_2^{(1)}(t_0), ..., y_1^{(n-1)}(t_0) + y_2^{(n-1)}(t_0)) \\
&= (y_1(t_0), ..., y^{(n-1)}(t_0)) + (y_2(t_0), ..., y_2^{(n-1)}(t_0)) \\
&= \varphi(y_1) + \varphi(y_2)
\end{aligned}$$

and that

$$\begin{aligned}
\varphi(cy_1) &= ((cy_1)(t_0), ..., (cy_1)^{(n-1)}(t_0)) \\
&= (cy_1(t_0), ..., cy_1^{(n-1)}(t_0)) \\
&= c(y_1(t_0), ..., y_1^{(n-1)}(t_0)) \\
&= c\varphi(y_1)
\end{aligned}$$

and hence this shows that φ is indeed a linear map. Next, suppose $y_1, y_2 \in S$ are such that

$$\varphi(y_1) = \varphi(y_2)$$

for some $y_1, y_2 \in S$. This implies that

$$(y_1(t_0), ..., y_1^{(n-1)}(t_0)) = (y_2(t_0), ..., y_2^{(n-1)}(t_0))$$

By the *uniqueness* condition in **Theorem 1**, it follows that $y_1 = y_2$, i.e they are the same solution. So, the map φ is one-one. Finally, suppose $(\alpha_0, ..., \alpha_{n-1}) \in \mathbb{R}^n$ is any point. By the *existence* part of **Theorem 1**, there is some $y \in S$ with

$$\varphi(y) = (\alpha_0, ..., \alpha_{n-1})$$

and hence this means that φ is a surjective map. So, it follows that φ is an isomorphism, i.e $S \cong \mathbb{R}^n$ as vector spaces. This completes the proof.

5). Consider the following equation

$$(*) \qquad \dot{\vec{\boldsymbol{x}}}(t) = A(t)\vec{\boldsymbol{x}}(t)$$

where $A(t)$ is a matrix as in the homework sheet. We show that the solutions to the equation $(*)$ form an n-dimensional real vector space. Let $t_0 \in (a, b)$ be any point, and let S be the space of solutions to $(*)$. We will produce a vector space isomorphism $\varphi : S \to \mathbb{R}^n$ as follows: for any $\vec{\boldsymbol{x}} \in S$, define

$$\varphi(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}}(t_0)$$

Suppose $\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2 \in S$ and let $c \in \mathbb{R}$. Then, it is clear that $\vec{\boldsymbol{x}}_1 + \vec{\boldsymbol{x}}_2$ and $c\vec{\boldsymbol{x}}_1$ are also solutions to $(*)$, because the derivative as an operator is a linear operator and because matrix multiplication is distributive over addition. Also observe that

$$\varphi(\vec{\boldsymbol{x}}_1 + \vec{\boldsymbol{x}}_2) = (\vec{\boldsymbol{x}}_1 + \vec{\boldsymbol{x}}_2)(t_0) = \vec{\boldsymbol{x}}_1(t_0) + \vec{\boldsymbol{x}}_2(t_0) = \varphi(\vec{\boldsymbol{x}}_1) + \varphi(\vec{\boldsymbol{x}}_2)$$

and that

$$\varphi(c\vec{\boldsymbol{x}}_1) = (c\vec{\boldsymbol{x}}_1)(t_0) = c\vec{\boldsymbol{x}}_1(t_0) = c\varphi(\vec{\boldsymbol{x}}_1)$$

and hence φ is a linear map indeed. Next, suppose $\varphi(\vec{\boldsymbol{x}}_1) = \varphi(\vec{\boldsymbol{x}}_2)$ for some $\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2 \in S$. This implies that $\vec{\boldsymbol{x}}_1$ and $\vec{\boldsymbol{x}}_2$ are solutions of $(*)$ with $\vec{\boldsymbol{x}}_1(t_0) = \vec{\boldsymbol{x}}_2(t_0)$. By the *uniqueness* condition in **Theorem 2** as mentioned in the homework sheet, it follows that $\vec{\boldsymbol{x}}_1 = \vec{\boldsymbol{x}}_2$, i.e they are the same map. This shows that φ is an injective map. Finally, if $\vec{\boldsymbol{\alpha}} \in \mathbb{R}^n$ is any point, then by the *existence* part of **Theorem 2** there is some $\vec{\boldsymbol{x}} \in S$ with $\varphi(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}}(t_0) = \vec{\boldsymbol{\alpha}}$, and this implies that φ is a surjective map. So, $S \cong \mathbb{R}^n$, and this completes our proof.

6). For this problem, let A be a constant matrix, and let $(a, b) = \mathbb{R}$. Then, our DE becomes

$$(\dagger) \qquad \dot{\vec{\boldsymbol{x}}}(r) = A\vec{\boldsymbol{x}}(r) \quad , \quad r \in \mathbb{R}$$

We want to show that $g(s + t) = g(s)g(t)$ for any $s, t \in \mathbb{R}$, where the right hand side is function composition. In particular, we want to show that for every $\vec{\boldsymbol{\alpha}} \in \mathbb{R}^n$, the equation

$$g(s + t)(\vec{\boldsymbol{\alpha}}) = g(s) \circ g(t)(\vec{\boldsymbol{\alpha}})$$

holds. To show this, let $\vec{\boldsymbol{x}}_0$ be the unique solution to $(\dagger)$ with $\vec{\boldsymbol{x}}_0(0) = \vec{\boldsymbol{\alpha}}$. So, we see that $g(s + t)(\vec{\boldsymbol{\alpha}}) = \vec{\boldsymbol{x}}_0(s + t)$. Also, we see that $g(t)(\vec{\boldsymbol{\alpha}}) = \vec{\boldsymbol{x}}_0(t)$, and hence

$$g(s) \circ g(t)(\vec{\boldsymbol{\alpha}}) = g(s)(\vec{\boldsymbol{x}}_0(t))$$

Let $\vec{\boldsymbol{x}}_1$ be the unique solution to $(\dagger)$ with $\vec{\boldsymbol{x}}_1(0) = \vec{\boldsymbol{x}}_0(t)$. So, it follows that $g(s)(\vec{\boldsymbol{x}}_0(t)) = \vec{\boldsymbol{x}}_1(s)$. Now, consider the map $\vec{\boldsymbol{x}}_2 : \mathbb{R} \to \mathbb{R}^n$ defined by

$$\vec{\boldsymbol{x}}_2(r) = \vec{\boldsymbol{x}}_0(r + t)$$

From the above definition, we see that

$$\dot{\vec{\boldsymbol{x}}}_2(r) = \dot{\vec{\boldsymbol{x}}}_0(r + t) = A\vec{\boldsymbol{x}}_0(r + t) = A\vec{\boldsymbol{x}}_2(r)$$

So, it follows that $\vec{\boldsymbol{x}}_2$ is a solution of $(\dagger)$ with $\vec{\boldsymbol{x}}_2(0) = \vec{\boldsymbol{x}}_0(t)$. So by uniqueness, it follows that $\vec{\boldsymbol{x}}_2 = \vec{\boldsymbol{x}}_1$, i.e

$$\vec{\boldsymbol{x}}_1(r) = \vec{\boldsymbol{x}}_0(r + t)$$

for every $r \in \mathbb{R}$. So, we see that

$$\vec{\boldsymbol{x}}_1(s) = \vec{\boldsymbol{x}}_0(s + t)$$

All of this implies that

$$g(s+t)(\vec{\alpha}) = g(s) \circ g(t)(\vec{\alpha})$$

Since $s, r \in \mathbb{R}$ and $\vec{\alpha} \in \mathbb{R}^n$ were arbitrary, it follows that

$$g(s+t) = g(s)g(t)$$

and this completes our proof.

Next, we will show that $g(0)$ is the identity map, i.e $g(0)(\vec{\alpha}) = \vec{\alpha}$ for every $\vec{\alpha} \in \mathbb{R}^n$. But this is easy to see: let $\vec{\alpha} \in \mathbb{R}^n$ be any point, and let $\vec{x}$ be the unique solution of (†) with $\vec{x}(0) = \vec{\alpha}$. Then, we have

$$g(0)(\vec{\alpha}) = \vec{x}(0) = \vec{\alpha}$$

and thus $g(0)$ is indeed the identity mapping. This completes our proof.

7). We will now show that each $g(s)$ is an invertible linear transformation. First, we show that $g(s)$ is a linear transformation. So, let $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^n$ be given and let $c \in \mathbb{R}$. Let $\vec{x}_\alpha, \vec{x}_\beta$ and $\vec{x}_{\alpha+\beta}$ be unique solutions of the DE (†) with that $\vec{x}_\alpha(0) = \vec{\alpha}$, $\vec{x}_\beta(0) = \vec{\beta}$ and $\vec{x}_{\alpha+\beta}(0) = \vec{\alpha} + \vec{\beta}$. Consider the map $\vec{x}_0$ on $\mathbb{R}$ given by

$$\vec{x}_{\mathbf{0}}(t) = \vec{x}_\alpha(t) + \vec{x}_\beta(t) \quad , \quad t \in \mathbb{R}$$

Then, note that $\vec{x}_0(0) = \vec{\alpha} + \vec{\beta}$, and that

$$\dot{\vec{x}}_0(t) = \dot{\vec{x}}_\alpha(t) + \dot{\vec{x}}_\beta(t) = A\vec{x}_\alpha(t) + A\vec{x}_\beta(t) = A\vec{x}_0(t)$$

and hence $\vec{x}_0$ is a solution of (†). So by uniqueness, it follows that $\vec{x}_0 = \vec{x}_{\alpha+\beta}$, i.e

$$\vec{x}_{\alpha+\beta} = \vec{x}_\alpha + \vec{x}_\beta$$

So, it follows that

$$g(s)(\vec{\alpha} + \vec{\beta}) = \vec{x}_{\alpha+\beta}(s) = \vec{x}_\alpha(s) + \vec{x}_\beta(s) = g(s)(\vec{\alpha}) + g(s)(\vec{\beta})$$

so that $g(s)$ preserves addition.

Now, let $\vec{x}_{c\alpha}$ be the unique solution of (†) with $\vec{x}_{c\alpha}(0) = c\alpha$. Again, let $\vec{x}_1$ be the map on $\mathbb{R}$ given by

$$\vec{x}_1(t) = c\vec{x}_\alpha(t) \quad , \quad t \in \mathbb{R}$$

Clearly, $\vec{x}_1(0) = c\vec{\alpha}$, and also

$$\dot{\vec{x}}_1(t) = c\dot{\vec{x}}_\alpha(t) = cA\vec{x}_\alpha(t) = A\vec{x}_1(t)$$

which implies that $\vec{x}_1$ is a solution of (†). So by uniquness, we see that $\vec{x}_1 = \vec{x}_{c\alpha}$, i.e

$$\vec{x}_{c\alpha} = c\vec{x}_\alpha$$

So, we see that

$$g(s)(c\vec{\alpha}) = \vec{x}_{c\alpha}(s) = c\vec{x}_\alpha(s) = cg(s)(\vec{\alpha})$$

and hence it follows that $g(s)$ in indeed a linear map.

Now, suppose $g(s)(\vec{\alpha}) = g(s)(\vec{\beta})$ for some $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^n$, and let the maps $\vec{x}_\alpha$, $\vec{x}_\beta$ have the same meaning as above. This implies that $\vec{x}_\alpha(s) = \vec{x}_\beta(s) = \vec{\gamma}$. Now, we know that $s \in \mathbb{R}$ and that both $\vec{x}_\alpha$ and $\vec{x}_\beta$ are solutions of (†). So, applying the *uniqueness* part **Theorem 2** with the initial condition $\vec{x}(s) = \vec{\gamma}$, it follows that $\vec{x}_\alpha = \vec{x}_\beta$. In particular, we have

$$\vec{\alpha} = \vec{x}_\alpha(0) = \vec{x}_\beta(0) = \vec{\beta}$$

and so this implies that $g(s)$ is an injective map.

To show surjectivity, we know by **Theorem 2** that for any $\vec{\boldsymbol{\alpha}} \in \mathbb{R}^n$ there is a solution $\vec{\boldsymbol{x}}$ of (†) such that $\vec{\boldsymbol{x}}(s) = \vec{\boldsymbol{\alpha}}$. Let $\gamma = \vec{\boldsymbol{x}}(0)$. In this case, we see that

$$g(s)(\vec{\gamma}) = \vec{\boldsymbol{x}}(s) = \vec{\boldsymbol{\alpha}}$$

and hence $g(s)$ is a surjective map. So, it follows that $g(s)$ is an invertible linear transformation, and this completes our proof.

8). We identify linear endomorphisms of $\mathbb{R}^n$ with $n \times n$ matrices, and let g be as above. We will show that

$$\lim_{t \to 0} \frac{g(t) - g(0)}{t} = A$$

We know that $g(t)$ is an invertible $n \times n$ matrix for every $t \in \mathbb{R}$. Moreover, the i^{th} column of the matrix $g(t)$ is given by $g(t)(\vec{\boldsymbol{e_i}})$, where $\vec{\boldsymbol{e_1}}, ..., \vec{\boldsymbol{e_n}}$ are the standard basis vectors of $\mathbb{R}^n$. So, let $1 \leq i \leq n$ be fixed. Let $\vec{\boldsymbol{x_i}}$ be the unique solution of (†) with $\vec{\boldsymbol{x_i}}(0) = \vec{\boldsymbol{e_i}}$. So, we see that

$$g(t)(\vec{\boldsymbol{e_i}}) = \vec{\boldsymbol{x_i}}(t) \quad , \quad t \in \mathbb{R}$$

Now suppose $\vec{\boldsymbol{x_i}} = (x_{1i}, ..., x_{ni})$ are the component functions of $\vec{\boldsymbol{x_i}}$, and hence the i^{th} column of $g(t)$ is

$$\begin{bmatrix} x_{1i}(t) \\ x_{2i}(t) \\ \cdots \\ x_{ni}(t) \end{bmatrix}$$

Also from the equality $\vec{\boldsymbol{x_i}}(0) = \vec{\boldsymbol{e_i}}$ we have

$$\begin{bmatrix} x_{1i}(0) \\ x_{2i}(0) \\ \cdots \\ x_{ii}(0) \\ \cdots \\ x_{ni}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ \cdots \\ 0 \end{bmatrix}$$

Moreover, writing equation (†) in coordinate form translates to

$$\begin{bmatrix} \dot{x}_{1i}(t) \\ \dot{x}_{2i}(t) \\ \cdots \\ \dot{x}_{ni}(t) \end{bmatrix} = A \begin{bmatrix} x_{1i}(t) \\ x_{2i}(t) \\ \cdots \\ x_{ni}(t) \end{bmatrix} \quad , \quad t \in \mathbb{R}$$

Putting $t = 0$ above, this means

$$\begin{bmatrix} \dot{x}_{1i}(0) \\ \dot{x}_{2i}(0) \\ \cdots \\ \dot{x}_{ni}(0) \end{bmatrix} = A \begin{bmatrix} x_{1i}(0) \\ x_{2i}(0) \\ \cdots \\ x_{ni}(0) \end{bmatrix} = A\vec{\boldsymbol{e_i}}$$

Writing the above equation in terms of limits, we see that

$$\lim_{t \to 0} \begin{bmatrix} \dfrac{x_{1i}(t) - x_{1i}(0)}{t} \\ \dfrac{x_{2i}(t) - x_{2i}(0)}{t} \\ \cdots \\ \dfrac{x_{ni}(t) - x_{ni}(0)}{t} \end{bmatrix} = \lim_{t \to 0} \frac{1}{t} \begin{bmatrix} x_{1i}(t) - x_{1i}(0) \\ x_{2i}(t) - x_{2i}(0) \\ \cdots \\ x_{ni}(t) - x_{ni}(0) \end{bmatrix} = \lim_{t \to 0} \frac{g(t)(\vec{\boldsymbol{e_i}}) - \vec{\boldsymbol{e_i}}}{t} = A\vec{\boldsymbol{e_i}}$$

Now, note that in the above equation $A\vec{\boldsymbol{e_i}}$ is the i^{th} column of the matrix A, and $\vec{\boldsymbol{e_i}}$ is the i^{th} column of the identity matrix I. So, because the last equation is true for all $1 \leq i \leq n$, it follows that

$$\lim_{t\to 0} \frac{g(t) - I}{t} = A$$

Also, we know that $g(0) = I$ since $g(0)$ is the identity map. So, we get

$$\lim_{t\to 0} \frac{g(t) - g(0)}{t} = A$$

and this completes the proof.

9). Consider the n-th order linear differential equation

$$(**) \qquad y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + ... + p_n(t)y(t) = 0$$

where each p_i is a continuous function on (a, b), and let $t_0 \in (a, b)$ be a point, and a set of initial conditions

$$y^{(i)}(t_0) = \alpha_i \quad , \quad 0 \leq i \leq n - 1$$

where $\vec{\boldsymbol{\alpha}} = (\alpha_0, ..., \alpha_{n-1}) \in \mathbb{R}^n$ is a point. We will show that this equation is equivalent to a first order linear $\mathbb{R}^n$-valued differential equation of the form $(*)$ on (a, b). To show this, consider the following matrix of functions: let

$$\vec{\boldsymbol{\lambda}}(t) = (p_n(t), p_{n-1}(t), ..., p_1(t)) \quad , \quad t \in (a, b)$$

and let $A(t)$ be the $n \times n$ matrix of functions given by

$$A(t) = \begin{bmatrix} \vec{\boldsymbol{e_2}} \\ \vec{\boldsymbol{e_3}} \\ ... \\ \vec{\boldsymbol{e_n}} \\ -\vec{\boldsymbol{\lambda}}(t) \end{bmatrix}$$

So, consider a DE of the form

$$\dot{\vec{\boldsymbol{x}}}(t) = A(t)\vec{\boldsymbol{x}}(t)$$

along with the initial condition $\vec{\boldsymbol{x}}(t_0) = \vec{\boldsymbol{\alpha}}$. Suppose $\vec{\boldsymbol{x}}$ is a solution of this DE and let $\vec{\boldsymbol{x}} = (x_1, ..., x_n)$ where each x_i is an $\mathbb{R}$-valued function on (a, b). We will show that x_1 is a solution of the DE $(**)$, and that will complete the proof.

Written in coordinate form, the differential equation above becomes the following.

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ ... \\ \dot{x}_{n-1}(t) \\ \dot{x_n}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_3(t) \\ ... \\ x_n(t) \\ -p_n(t)x_1(t) - ... - p_1(t)x_n(t) \end{bmatrix} \quad , \quad t \in (a, b)$$

So, it follows that $x_i(t) = x_1^{(i-1)}(t)$ for each $2 \leq i \leq n$ and $t \in (a, b)$, and also

$$\dot{x_n}(t) = x_1^{(n)}(t) = -p_n(t)x_1(t) - p_{n-1}(t)x_2(t) - ... - p_1(t)x_n(t)$$

and the above equation implies that

$$x_1^{(n)}(t) + p_1(t)x^{(n-1)}(t) + ... + p_n(t)x_1(t) = 0$$

and so x_1 is a solution of the DE $(**)$. Also, the initial condition in coordinate form is written as

$$(x_1(t_0), ..., x_n(t_0)) = (\alpha_0, ..., \alpha_{n-1})$$

and this implies that $x_1^{(i)}(t_0) = \alpha_i$ for each $1 \leq i \leq n-1$.

Conversely, if the function x_1 is the solution to $(**)$ along with the condition $x_1^{(i)}(t_0) = \alpha_i$ for $1 \leq i \leq n-1$, then by putting $\vec{\boldsymbol{x}} = (x_1, x_1^{(1)}, ..., x_1^{(n-1)})$ and reversing all the steps we did above, we see that $\vec{\boldsymbol{x}}$ satisfies the equation

$$\dot{\vec{\boldsymbol{x}}}(t) = A(t)\vec{\boldsymbol{x}}(t) \quad , \quad t \in (a, b)$$

where $A(t)$ is the matrix of functions constructed above, and that $\vec{\boldsymbol{x}}(t_0) = \vec{\boldsymbol{\alpha}}$. So, this shows the uniqueness of the solution to $(**)$, because **Theorem 2** guarantees the uniqueness of the solution to $(*)$.

So, it follows that **Theorem 1** is a special case of **Theorem 2**, if we take the map $g(t)$ in **Theorem 1** to be identically zero.