HW-3

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1). In this problem we will solve problems **13** and **22** from Cookbook-I.

13 $(e^x \sin y - 2y \sin x)\mathrm{d}x + (e^x \cos y + 2\cos x)\mathrm{d}y = 0$

Solution. This equation is equivalent to

$$(e^x \sin y - 2y \sin x) + (e^x \cos y + 2\cos x)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

We now show that this is an exact equation. To see this, observe that

$$\frac{\partial(e^x \sin y - 2y \sin x)}{\partial y} = e^x \cos y - 2 \sin x$$

and that

$$\frac{\partial(e^x \cos y + 2\cos x)}{\partial x} = e^x \cos y - 2 \sin x$$

and the above two equations show that our DE is exact. Now, we need to find P such that

$$\frac{\partial P}{\partial x} = e^x \sin y - 2y \sin x \quad , \quad \frac{\partial P}{\partial y} = e^x \cos y + 2\cos x \tag{0.1}$$

So, consider

$$\begin{aligned} P(x,y) &= \int e^x \sin y - 2y \sin x \, \mathrm{d}x \\ &= e^x \sin y + 2y \cos x + g(y) \end{aligned}$$

for some differentiable function g of y. Clearly, differentiating $P(x,y)$ with respect to x, we see that $P(x,y)$ satisifies the first half of equation (0.1). Now, differentiating $P(x,y)$ with respect to y and equating it to the RHS of the second half of equation (0.1) we get

$$\frac{\partial P}{\partial y} = e^x \cos y + 2\cos x + g'(y) = e^x \cos y + 2 \cos x$$

so that $g'(y) = 0$, i.e $g(y) = K$ for some $K \in \mathbb{R}$. We can take $K = 0$. So, the solution to our DE is given by the implicit equation

$$P(x,y) = e^x \sin y + 2y \cos x = C$$

for some $C \in \mathbb{R}$. ■

22 $2y^2 y'' + 2y(y')^2 = 1$

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Solution. We assume that $y > 0$ at all points where it is defined. Dividing throughout by $2y^2$, this DE becomes

$$y'' = \frac{1}{2y^2} - \frac{(y')^2}{y}$$

This is a second order equation with the independent variable missing. Putting $v = y'$, we see that

$$v\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{1}{2y^2} - \frac{v^2}{y}$$

which on dividing throughout by v, we rewrite as

$$\frac{\mathrm{d}v}{\mathrm{d}y} + \frac{v}{y} = \frac{1}{2y^2}v^{-1}$$

This is an example of a Bernoulli equation. We substitute $u = v^{1-(-1)} = v^2$ and get

$$\frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{u}{y} = \frac{1}{2y^2}$$

This is a first order linear DE. The integrating factor is

$$\mu(y) = \exp\int\frac{2}{y}\,\mathrm{d}y = \exp\left(2\ln y + K\right) = e^K y^2 = K'y^2$$

where $K' \in \mathbb{R}$. So, the solution to the linear DE is

$$u = \frac{1}{K'y^2}\int K'y^2\frac{1}{y^2}\,\mathrm{d}y = \frac{1}{y^2}\int\mathrm{d}y = \frac{y+C}{y^2}$$

for some $C \in \mathbb{R}$. So, it follows that

$$v^2 = \frac{y+C}{y^2}$$

and hence

$$v = \sqrt{\frac{y+C}{y^2}} = \frac{\sqrt{y+C}}{y}$$

So, we just have to solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{y+C}}{y}$$

which is clearly a separable DE. We get

$$\frac{y}{\sqrt{y+C}}\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

Integrating both sides with respect to x, we get

$$\int\frac{y}{\sqrt{y+C}}\,\mathrm{d}y = \int\mathrm{d}x = x + C_0$$

for some $C_0 \in \mathbb{R}$. The integral on the LHS can be solved by substituting $t = y + C$, and we get

$$\int\frac{y}{\sqrt{y+C}}\,\mathrm{d}y = \int\frac{t-C}{\sqrt{t}}\,\mathrm{d}t = \int\sqrt{t}\,\mathrm{d}t - C\int\frac{1}{\sqrt{t}}\,\mathrm{d}t = \frac{2t^{\frac{3}{2}}}{3} - 2Ct^{\frac{1}{2}} + C'$$

for some $C' \in \mathbb{R}$. So, we get

$$\int\frac{y}{\sqrt{y+C}}\,\mathrm{d}y = \frac{2(y+C)^{\frac{3}{2}}}{3} - 2C(y+C)^{\frac{1}{2}} + C'$$

and combining all this, we see that the solution to our original DE is

$$\frac{2(y+C)^{\frac{3}{2}}}{3} - 2C(y+C)^{\frac{1}{2}} + C' = x + C_0$$

which can be rewritten as

$$\frac{2(y+C)^{\frac{3}{2}}}{3} - 2C(y+C)^{\frac{1}{2}} = x + K$$

where $K = C_0 - C' \in \mathbb{R}$. ■

2). In this problem we will solve problems **3** and **9** from Cookbook-II.

3 $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 6\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - 10y = 0$, $y(0) = 2$, $y'(0) = -7$ and $y''(0) = 47$

Solution. The characteristic polynomial of the DE is

$$t^3 + 6t^2 + 3t - 10 = (t-1)(t+2)(t+5)$$

and hence this has three distinct roots, each of multiplicity 1. So, the space of solutions has basis e^x, e^{-2x} and e^{-5x}, and hence the general solution is given by

$$y = c_1e^x + c_2e^{-2x} + c_3e^{-5x}$$

We can calculate the constants $c_1, c_2, c_3 \in \mathbb{R}$ using the given initial conditions. The condition $y(0) = 2$ gives us

$$c_1 + c_2 + c_3 = 2$$

The condition $y'(0) = -7$ gives

$$c_1 - 2c_2 - 5c_3 = -7$$

and the condition $y''(0) = 47$ gives us

$$c_1 + 4c_2 + 25c_3 = 47$$

and solving this system of linear equations in three variables, we get

$$(c_1, c_2, c_3) = (1, -1, 2)$$

and hence

$$y = e^x - e^{-2x} + 2e^{-5x}$$

■

9 $y^{(6)} - 3y^{(5)} + 40y^{(3)} - 180y'' + 324y' - 432y = 0$ [Hint: $1 + i\sqrt{5}$ is a root (with positive multiplicity) of the characteristic polynomial.]

Solution. The characteristic polynomial of the DE is

$$t^6 - 3t^5 + 40t^3 - 180t^2 + 324t - 432$$

As given in the hint, $1 + i\sqrt{5}$ is a root, and because it occurs as a conjugate pair we see that $(t - 1 - i\sqrt{5})(t - 1 + i\sqrt{5}) = (t-1)^2 + 5$ is a factor of the given polynomial. Using long division, we get

$$t^6 - 3t^5 + 40t^3 - 180t^2 + 324t - 432 = ((t-1)^2 + 5)(t^4 - t^3 - 8t^2 + 30t - 72)$$

Now, it can be checked that $3, -4$ are roots of the biquadratic factor, i.e $(t-3)(t+4) = t^2 + t - 12$ is a factor of the above biquadratic factor. Again by long division we see that

$$t^4 - t^3 - 8t^2 + 30t - 72 = (t^2 + t - 12)(t^2 - 2t + 6)$$

Finally, the polynomial t^2-2t+6 has $1+i\sqrt{5}, 1-i\sqrt{5}$ as its roots. So, it follows that our original polynomial $t^6-3t^5+40t^3-180t^2+324t-432$ has the following roots:

$$\begin{aligned} 1+i\sqrt{5} &\text{ with multiplicity } 2 \\ 1-i\sqrt{5} &\text{ with multiplicity } 2 \\ 3 &\text{ with multiplicity } 1 \\ -4 &\text{ with multiplicity } 1 \end{aligned}$$

So, it follows that the basis of the space of solutions consists of

$$e^{3x}, e^{-4x}, e^x\cos(\sqrt{5}x), xe^x\cos(\sqrt{5}x), e^x\sin(\sqrt{5}x), xe^x\sin(\sqrt{5}x)$$

So, the general solution to this DE is given by

$$y = c_1e^{3x}+c_2e^{-4x}+c_3e^x\cos(\sqrt{5}x)+c_4xe^x\cos(\sqrt{5}x)+c_5e^x\sin(\sqrt{5}x)+c_6xe^x\sin(\sqrt{5}x)$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{R}$ are constants. ■

3). Let I be an interval in $\mathbb{R}$ and let

$$(*) \qquad \dot{\vec{\boldsymbol{x}}}(t) = A(t)\vec{\boldsymbol{x}}(t) \qquad (t \in I)$$

be a linear first order DE where A is an $n \times n$ matrix of continuous functions on I. Suppose $\vec{\boldsymbol{y}}_1(t), ..., \vec{\boldsymbol{y}}_n(t)$ are n solutions of the above DE on I. Let

$$W = W(\vec{\boldsymbol{y}}_1, ..., \vec{\boldsymbol{y}}_n) : I \to \mathbb{R}$$

be the map given by

$$W(t) = \det[\vec{\boldsymbol{y}}_1(t), ..., \vec{\boldsymbol{y}}_n(t)] \qquad (t \in I)$$

We show that either W is identically zero on I or it is nowhere vanishing on I. The main idea that we will be using is the *uniqueness* of solutions to $(*)$, which has been proven in Lecture-7.

We need to show that if $W(t_0) = 0$ for some $t_0 \in I$, then $W(t) = 0$ for all $t \in I$. So, take such a $t_0 \in I$. This means that

$$\det[\vec{\boldsymbol{y}}_1(t_0), ..., \vec{\boldsymbol{y}}_n(t_0)] = 0$$

and hence the vectors $\vec{\boldsymbol{y}}_1(t_0), ..., \vec{\boldsymbol{y}}_n(t_0)$ are linearly dependent. So, there are constants $a_1, ..., a_n \in \mathbb{R}$ not all zero such that

$$a_1\vec{\boldsymbol{y}}_1(t_0) + ... + a_n\vec{\boldsymbol{y}}_n(t_0) = 0$$

Without loss of generality, suppose $a_1 \neq 0$. So, we see that

$$\vec{\boldsymbol{y}}_1(t_0) = -\frac{a_2}{a_1}\vec{\boldsymbol{y}}_2(t_0) - ... - \frac{a_n}{a_1}\vec{\boldsymbol{y}}_n(t_0)$$

i.e $\vec{\boldsymbol{y}}_1(t_0)$ is a linear combination of $\vec{\boldsymbol{y}}_2(t_0), ..., \vec{\boldsymbol{y}}_n(t_0)$. Put $\vec{\boldsymbol{a}}_0 = \vec{\boldsymbol{y}}_1(t_0)$ and consider the map

$$-\frac{a_2}{a_1}\vec{\boldsymbol{y}}_2 - ... - \frac{a_n}{a_1}\vec{\boldsymbol{y}}_n = \vec{\boldsymbol{y}} : I \to \mathbb{R}^n$$

Because $\vec{\boldsymbol{y}}_2, ..., \vec{\boldsymbol{y}}_n$ are solutions of $(*)$, $\vec{\boldsymbol{y}}$ being a linear combination is also a solution of $(*)$. Moreover, we see that

$$\vec{\boldsymbol{y}}(t_0) = \vec{\boldsymbol{y}}_1(t_0) = \vec{\boldsymbol{a}}_0$$

So, by the *uniqueness* of solutions of $(*)$ on the initial condition $\vec{\boldsymbol{x}}(t_0) = \vec{\boldsymbol{a}}_0$, it follows that $\vec{\boldsymbol{y}} = \vec{\boldsymbol{y}}_1$ on I. In other words, this means that $\vec{\boldsymbol{y}}_1(t)$ is a linear combination of $\vec{\boldsymbol{y}}_2(t), ..., \vec{\boldsymbol{y}}_n(t)$ for each $t \in I$, and hence

$$W(t) = \det[\vec{\boldsymbol{y}}_1(t), ..., \vec{\boldsymbol{y}}_n(t)] = 0$$

for each $t \in I$, implying that W is identically zero on I. So, it follows that either W is identically zero on I, or W vanishes at no point of I, and this is what the claim was.

4). Let I be an interval in $\mathbb{R}$, and let $A : I \to M_{m,k}(\mathbb{R})$ and $B : I \to M_{k,n}(\mathbb{R})$ be differentiable on I. We will show that the map $t \mapsto A(t)B(t)$ gives us a differentiable map $AB : I \to M_{m,n}(\mathbb{R})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(A(t)B(t)\right) = \dot{A}(t)B(t) + A(t)\dot{B}(t) \qquad (t \in I)$$

Note that it is enough to show that for any $t_0 \in I$,

$$\lim_{h\to 0}\frac{A(t_0+h)B(t_0+h) - A(t_0)B(t_0)}{h} = \dot{A}(t_0)B(t_0) + A(t_0)\dot{B}(t_0)$$

where the above limit is taken with respect to the operator norm (or any other norm, since all are equivalent), and it is interpreted as a one-sided limit if t_0 is a boundary point of I. Showing this is a straightforward computation.

$$\begin{aligned}
&\lim_{h\to 0}\frac{A(t_0+h)B(t_0+h) - A(t_0)B(t_0)}{h} \\
&= \lim_{h\to 0}\frac{A(t_0+h)B(t_0+h) - A(t_0)B(t_0+h) + A(t_0)B(t_0+h) - A(t_0)B(t_0)}{h} \\
&= \lim_{h\to 0}\frac{B(t_0+h)[A(t_0+h) - A(t_0)] + A(t_0)[B(t_0+h) - B(t_0)]}{h} \\
&= \lim_{h\to 0}\frac{B(t_0+h)[A(t_0+h) - A(t_0)]}{h} + \lim_{h\to 0}\frac{A(t_0)[B(t_0+h) - B(t_0)]}{h} \\
&= \dot{A}(t_0)B(t_0) + A(t_0)\dot{B}(t_0)
\end{aligned}$$

where in the last step we have just split the limits into products of limits (the fact that this can be done was proven in the Analysis-2 course). This completes the proof.