HW-4

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1). $x^3 \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2xy = 2$, $x > 0$

Solution. Let our interval be $(0, \infty)$. First, we convert the given DE to a first order vector valued DE. Observe that the DE is equivalent to the following system of equations

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = y_2$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}x} = \frac{2y_1}{x^2} + \frac{2}{x^3}$$

i.e if $\vec{\boldsymbol{y}} = (y_1, y_2)$ is a particular solution to the above vector valued DE on the interval $(0, \infty)$, then its first coordinate, namely y_1 is a particular solution of our original DE. Note that this vector valued DE can be written as

$$\dot{\vec{\boldsymbol{y}}} = \begin{bmatrix} 0 & 1 \\ \frac{2}{x^2} & 0 \end{bmatrix} \vec{\boldsymbol{y}} + \begin{bmatrix} 0 \\ \frac{2}{x^3} \end{bmatrix} \tag{0.1}$$

Let $\vec{\boldsymbol{g}}(x) = \left(0, \frac{2}{x^3}\right)$ on $(0, \infty)$, so that $\vec{\boldsymbol{g}}$ is continuous. So, our vector valued DE is actually a first order linear DE, and this can be easily solved using variation of parameters. First, we need to find linearly independent solutions to the homogeneous DE

$$\dot{\vec{\boldsymbol{y}}} = \begin{bmatrix} 0 & 1 \\ \frac{2}{x^2} & 0 \end{bmatrix} \vec{\boldsymbol{y}} \tag{0.2}$$

which amounts to solving the system of equations

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = y_2 \tag{0.3}$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}x} = \frac{2y_1}{x^2} \tag{0.4}$$

and this is just the DE

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{2y}{x^2} = 0 \tag{0.5}$$

which is the same as

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2y = 0$$

Putting $t = \ln x$, the above equation is transformed as

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} - 2y = 0$$

The characteristic polynomial of this DE is

$$s^2 - s - 2 = (s+1)(s-2)$$

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and hence a pair of linearly independent solutions for this DE is $\{e^{-t}, e^{2t}\}$. So, a pair of linearly independent solutions for the DE (0.5) is $\{x^{-1}, x^2\}$. So, it follows that a pair of linearly independent solutions for the vector valued DE (0.2) is $\{\vec{\varphi}_1, \vec{\varphi}_2\}$ where

$$\vec{\varphi}_1(x) = \left(\frac{1}{x}, -\frac{1}{x^2}\right)$$
$$\vec{\varphi}_2(x) = \left(x^2, 2x\right)$$

and because the space of solutions of (0.2) is two-dimensional, it follows that $\vec{\varphi}_1, \vec{\varphi}_2$ constitute a basis. Now as done in class, put

$$M = \begin{bmatrix}\vec{\varphi}_1 & \vec{\varphi}_2\end{bmatrix} = \begin{bmatrix}\frac{1}{x} & x^2 \\ -\frac{1}{x^2} & 2x\end{bmatrix}$$

and put

$$\vec{u}(x) = \int (M^{-1}\vec{g})(x)\mathrm{d}x = \int \frac{1}{3}\begin{bmatrix}2x & -x^2 \\ \frac{1}{x^2} & \frac{1}{x}\end{bmatrix}\vec{g}(x)\mathrm{d}x = \int \frac{1}{3}\begin{bmatrix}\frac{-2}{x} \\ \frac{2}{x^4}\end{bmatrix}\mathrm{d}x = \frac{1}{3}\begin{bmatrix}-2\ln x \\ \frac{-2}{3x^3}\end{bmatrix} + \vec{a}_0$$

where $\vec{a}_0 = (C_1, C_2) \in \mathbb{R}^2$ is a constant vector. So, a solution of the DE (0.1) is given by

$$\vec{\psi}(x) = M(x)\vec{u}(x) = \frac{1}{3}\begin{bmatrix}\frac{-6\ln x - 2}{3x} \\ \frac{6\ln x - 4}{3x^2}\end{bmatrix} + \begin{bmatrix}\frac{C_1}{x} + C_2x^2 \\ \frac{-C_1}{x^2} + 2C_2x\end{bmatrix}$$

So, a general solution of our original DE is the first coordinate of $\vec{\psi}$, which is

$$y = \frac{-6\ln x - 2}{9x} + \frac{C_1}{x} + C_2x^2$$

where $C_1, C_2 \in \mathbb{R}$ are constants. To get a particular solution, we can just let $C_1 = C_2 = 0$. ∎

2). $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\dfrac{\mathrm{d}y}{\mathrm{d}t} + y = \dfrac{e^t}{1+t^2}$

Solution. We will follow a very similar strategy as in problem **1).** Let our interval be $\mathbb{R}$. First, we convert the given DE to a first order vector valued DE. Our DE is equivalent to the following system of equations

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = y_2$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = 2y_2 - y_1 + \frac{e^t}{1+t^2}$$

So, if $\vec{y} = (y_1, y_2)$ is a solution to the above vector valued DE on $\mathbb{R}$, then its first coordinate, namely y_1 is a solution of our original DE. The given vector valued DE can be written as

$$\dot{\vec{y}} = \begin{bmatrix}0 & 1 \\ -1 & 2\end{bmatrix}\vec{y} + \begin{bmatrix}0 \\ \frac{e^t}{1+t^2}\end{bmatrix} \tag{0.6}$$

Let $\vec{g}(t) = \left(0, \frac{e^t}{1+t^2}\right)$ on $\mathbb{R}$, so that $\vec{g}$ is continuous. We will now solve this vector valued DE using variation of parameters. So first consider the homogeneous DE

$$\dot{\vec{y}} = \begin{bmatrix}0 & 1 \\ -1 & 2\end{bmatrix}\vec{y} \tag{0.7}$$

and this amounts to solving the system of equatios

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = y_2 \tag{0.8}$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = 2y_2 - y_1 \tag{0.9}$$

which is just solving the DE

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0 \tag{0.10}$$

The characteristic polynomial of this DE is

$$s^2 - 2s + 1 = (s-1)^2$$

and hence a pair of linearly independent solutions is $\{e^t, te^t\}$. So, it follows that a pair of linearly independent solutions for the vector valued DE (0.7) is $\{\vec{\varphi}_1, \vec{\varphi}_2\}$, where

$$\vec{\varphi}_1(t) = \left(e^t, e^t\right)$$
$$\vec{\varphi}_2(t) = \left(te^t, e^t + te^t\right)$$

and because the space of solutions of (0.7) is two-dimensional, it follows that $\vec{\varphi}_1$, $\vec{\varphi}_2$ constitute a basis. Now as usual we put

$$M = \begin{bmatrix}\vec{\varphi}_1 & \vec{\varphi}_2\end{bmatrix} = \begin{bmatrix} e^t & te^t \\ e^t & e^t + te^t \end{bmatrix}$$

and we put

$$\vec{u}(t) = \int (M^{-1}\vec{g})(t)\mathrm{d}t = \int e^{-t}\begin{bmatrix} t+1 & -t \\ -1 & 1\end{bmatrix}\vec{g}(t)\mathrm{d}t = \int \begin{bmatrix} \frac{-t}{1+t^2} \\ \frac{1}{1+t^2}\end{bmatrix}\mathrm{d}t = \begin{bmatrix} -\frac{1}{2}\ln{(t^2+1)} \\ \arctan t\end{bmatrix} + \vec{a}_0$$

where $\vec{a}_0 = (C_1, C_2) \in \mathbb{R}^2$ is a constant vector. Since we are interested in a particular solution, we can let $\vec{a}_0 = \vec{0}$. So, a solution of the DE (0.6) is given by

$$\vec{\psi}(t) = M(t)\vec{u}(t) = \begin{bmatrix} \frac{-e^t\ln{(t^2+1)}}{2} + te^t \arctan t \\ \frac{-e^t\ln{(t^2+1)}}{2} + (e^t + te^t)\arctan t\end{bmatrix}$$

So, a particular solution of our original DE is the first coordinate of $\vec{\psi}(t)$, i.e a particular solution is

$$y = \frac{-e^t\ln{(t^2+1)}}{2} + te^t \arctan t$$

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3). $(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = x\sqrt{1-x^2}$, $0 < x < 1$

Solution. Let our interval be $(0, 1)$. First, we convert the given DE to a first order vector valued DE. Our DE is equivalent to the following system of equations

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = y_2$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}x} = \frac{y_2}{x(1-x^2)} + \frac{x}{\sqrt{1-x^2}}$$

So, if $\vec{\boldsymbol{y}} = (y_1, y_2)$ is a solution to the above vector valued DE on $(0,1)$, then its first coordinate, namely y_1 is a solution of our original DE. The given vector valued DE can be written as

$$\dot{\vec{\boldsymbol{y}}} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{x(1-x^2)} \end{bmatrix} \vec{\boldsymbol{y}} + \begin{bmatrix} 0 \\ \frac{x}{\sqrt{1-x^2}} \end{bmatrix} \tag{0.11}$$

Let $\vec{\boldsymbol{g}}(x) = \left(0, \frac{x}{\sqrt{1-x^2}}\right)$ on $(0,1)$, so that $\vec{\boldsymbol{g}}$ is continuous. We will now solve this vector valued DE using variation of parameters. So first consider the homogeneous DE

$$\dot{\vec{\boldsymbol{y}}} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{x(1-x^2)} \end{bmatrix} \vec{\boldsymbol{y}} \tag{0.12}$$

and this amounts to solving the system of equatios

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = y_2 \tag{0.13}$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}x} = \frac{y_2}{x(1-x^2)} \tag{0.14}$$

which is just solving the DE

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{1}{x(1-x^2)}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{0.15}$$

This is an example of a DE where the dependent variable is missing. Clearly, one solution to this DE is $y \equiv 1$ on $(0,1)$. To get another linearly independent solution, we put $v = y'$ in the above equation and get

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x(1-x^2)}v \implies \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x(1-x^2)}$$

Integrating both sides with respect to x, we get

$$\int \frac{1}{v}\mathrm{d}v = \int \frac{1}{x(1-x^2)}\mathrm{d}x = \int \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}\mathrm{d}x = \ln x - \frac{1}{2}\ln(1-x^2) + K$$

for some constant $K \in \mathbb{R}$. This gives us

$$\ln v = \ln x - \frac{1}{2}\ln(1-x^2) + K$$

which means

$$v = e^{\ln x - \frac{1}{2}\ln(1-x^2) + K} = K'\frac{x}{\sqrt{1-x^2}}$$

for some $K' \in \mathbb{R}$. So, integrating again with respect to x, we see that

$$y = \int v \, \mathrm{d}x = -K'\sqrt{1-x^2} + C$$

where $C \in \mathbb{R}$ is some constant. Since we need a basis element, we can simply let $K' = 1$ and $C = 0$. So, the two linearly independent solutions of (0.15) that we get are $\{1, -\sqrt{1-x^2}\}$ on $(0,1)$. So, it follows that a pair of linearly independent solutions for the vector valued DE (0.12) is $\{\vec{\boldsymbol{\varphi}}_1, \vec{\boldsymbol{\varphi}}_2\}$, where

$$\vec{\boldsymbol{\varphi}}_1(x) = (1, 0)$$

$$\vec{\boldsymbol{\varphi}}_2(x) = \left(-\sqrt{1-x^2}, \frac{x}{\sqrt{1-x^2}}\right)$$

and because the space of solutions of (0.12) is two-dimensional, it follows that $\vec{\varphi}_1$, $\vec{\varphi}_2$ constitute a basis. Now as usual we put

$$M = \begin{bmatrix}\vec{\varphi}_1 & \vec{\varphi}_2\end{bmatrix} = \begin{bmatrix}1 & -\sqrt{1-x^2} \\ 0 & \frac{x}{\sqrt{1-x^2}}\end{bmatrix}$$

and we put

$$\begin{aligned}\vec{u}(x) &= \int (M^{-1}\vec{g})(x)\mathrm{d}x \\ &= \int \begin{bmatrix}1 & \frac{1-x^2}{x} \\ 0 & \frac{\sqrt{1-x^2}}{x}\end{bmatrix} \vec{g}(x)\mathrm{d}x \\ &= \int \begin{bmatrix}\sqrt{1-x^2} \\ 1\end{bmatrix} \mathrm{d}x \\ &= \begin{bmatrix}\frac{1}{2}(x\sqrt{1-x^2} + \arcsin x) \\ x\end{bmatrix} + \vec{a}_0\end{aligned}$$

where $\vec{a}_0 = (C_1, C_2) \in \mathbb{R}^2$ is a constant vector. Since we are interested in a particular solution, we can let $\vec{a}_0 = \vec{0}$. So, a solution of the DE (0.11) is given by

$$\vec{\psi}(x) = M(x)\vec{u}(x) = \begin{bmatrix}\frac{1}{2}(\arcsin x - x\sqrt{1-x^2}) \\ \frac{x^2}{\sqrt{1-x^2}}\end{bmatrix}$$

So, a particular solution of our original DE is the first coordinate of $\vec{\psi}(x)$, i.e a particular solution is

$$y = \frac{1}{2}(\arcsin x - x\sqrt{1-x^2})$$

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4). Suppose $I = [t_0, t_1]$ and let a, b be continuous real valued functions on I. Let $p(t) = \int_{t_0}^{t} a(s)ds$. Suppose $u : I \to \mathbb{R}$ is $\mathscr{C}^1$ and satisfies the inequalities

$$\begin{aligned}\dot{u}(t) &\leq a(t)u(t) + b(t) \\ u(t_0) &= u_0\end{aligned}$$

for all $t \in I$. We will show that

$$u(t) \leq u_0 e^{p(t)} + \int_{t_0}^{t} e^{p(t)-p(s)} b(s)ds$$

Observe that we have

$$\dot{u}(t) - a(t)u(t) \leq b(t)$$

for each $t \in I$. Also, by the fundamental theorem of calculus, we know that $p'(t) = a(t)$ for all $t \in I$. So, multipliying both sides of the above inequality by the integrating factor $e^{-p(t)}$, we see that

$$\dot{u}(t)e^{-p(t)} - a(t)e^{-p(t)}u(t) \leq e^{-p(t)}b(t) \quad , \quad t \in I$$

Note that this inequality can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t)e^{-p(t)}) \leq e^{-p(t)}b(t) \quad , \quad t \in I$$

Applying the operator $\int_{t_0}^t \_\_ds$ (where $t \in I$) to both sides, we get

$$u(t)e^{-p(t)} - u(t_0)e^{-p(t_0)} \le \int_{t_0}^t e^{-p(s)}b(s)\mathrm{d}s$$

which means

$$u(t)e^{-p(t)} \le u_0 + \int_{t_0}^t e^{-p(s)}b(s)\mathrm{d}s$$

and multiplying both sides by $e^{p(t)}$ we get

$$u(t) \le u_0e^{p(t)} + \int_{t_0}^t e^{p(t)-p(s)}b(s)\mathrm{d}s$$

and this is what we wanted to show.

5). Let $I = [t_0, t_1]$ and φ, ψ and α continuous functions on I with $\alpha \ge 0$. Suppose

$$\varphi(t) \le \psi(t) + \int_{t_0}^t \alpha(s)\varphi(s)\mathrm{d}s \qquad (t \in I) \tag{0.16}$$

Let $q(t) = \int_{t_0}^t \alpha(s)ds$. We show that

$$\varphi(t) \le \psi(t) + \int_{t_0}^t e^{q(t)-q(s)}\alpha(s)\psi(s)\mathrm{d}s$$

First, we multiply both sides of the inequality (0.16) by $\alpha(t)$ (possible because $\alpha \ge 0$) to get

$$\alpha(t)\varphi(t) \le \alpha(t)\psi(t) + \alpha(t)h(t) \qquad (t \in I) \tag{0.17}$$

where h is defined on I by

$$h(t) = \int_{t_0}^t \alpha(s)\varphi(s)\mathrm{d}s \qquad (t \in I)$$

With this definition. inequality (0.17) can be rewritten as

$$h'(t) \le \alpha(t)\psi(t) + \alpha(t)h(t) \qquad (t \in E)$$

and note that $h(t_0) = 0$. So, by problem **4).** we get

$$h(t) \le \int_{t_0}^t e^{q(t)-q(s)}\alpha(s)\psi(s)\mathrm{d}s \qquad (t \in I)$$

and this means that

$$\int_{t_0}^t \alpha(s)\varphi(s)\mathrm{d}s \le \int_{t_0}^t e^{q(t)-q(s)}\alpha(s)\psi(s)\mathrm{d}s$$

Using this information in the given inequality (0.16) we get

$$\varphi(t) \le \psi(t) + \int_{t_0}^t e^{q(t)-q(s)}\alpha(s)\psi(s)\mathrm{d}s$$

and this proves the claim.

6). Let Ω, f, t_0, I, u and v be as in the problem statement. We will show that

$$u(t) \leq v(t) \quad , \quad t \in I$$

For the sake of contradiction, suppose $u(t_2) > v(t_2)$ for some $t_2 \in I$. Clearly, we have the inequality $t_0 < t_2$.

Consider the function $u - v$, which is clearly $\mathscr{C}^1$ on I. Because $u(t_0) - v(t_0) \leq 0$ and $u(t_2) - v(t_2) > 0$, the **Intermediate Value Theorem** implies that $u(t') - v(t') = 0$ for some $t' \in [t_0, t_2)$. This is to say that the set

$$X := \{t \in [t_0, t_2) \mid u(t) - v(t) = 0\}$$

is *non-empty*. Clearly, X is also bounded above. So, let $t_1 = \sup X$, and it is clear that $t_1 \in [t_0, t_2]$. We claim that $t_1 < t_2$. But this is easy to see, because $u(t_2) - v(t_2) > 0$ and because $u - v$ is a continuous function, there is some open interval around the point t_2 contained in I such that $u - v$ is positive in this open interval. Also, the continuity of $u - v$ implies that $u(t_1) - v(t_1) = 0$.

Now, consider the interval $J = [t_1, t_2]$, and above, by our choice of t_1, we have shown that $u(t_1) = v(t_1)$ and $u(t) - v(t) > 0$ for every $t \in J \setminus \{t_1\}$. Now, for every $t \in J$, we see that

$$\begin{aligned} \dot{u}(t) - \dot{v}(t) &= \dot{u}(t) - f(t, v(t)) \\ &\leq f(t, u(t)) - f(t, v(t)) \\ &\leq |f(t, u(t)) - f(t, v(t))| \\ &\leq L|u(t) - v(t)| \\ &= L(u(t) - v(t)) \end{aligned} \tag{‡}$$

and here we have used the fact that f is Lipschitz continuous in the second variable on Ω. To summarise, we have the following three conditions on $J = [t_1, t_2]$:

(1) $(u - v)(t_1) = 0$.
(2) $u - v > 0$ on $J \setminus \{t_1\}$.
(3) $(u \dot{-} v) \leq L(u - v)$ on J.

With points (1) and (3) in mind, we apply problem **4).** and get

$$(u - v)(t) \leq 0 \quad , \quad t \in J$$

which is a contradiction to point number (2) above. So, it follows that $u(t) \leq v(t)$ for all $t \in I$, and this completes the proof.

7). The proof of this result in the new set of hypothesis is exactly the same as in problem **6).** There is only one new change, which I will now mention. The construction of the interval $J = [t_1, t_2]$ remains the same. So, it is still true that $u(t_1) = v(t_1)$ and that $u - v > 0$ on $J \setminus \{t_1\}$. Infact, the chain of reasoning in (‡) is also valid; note that $u(t) \geq v(t)$ for every $t \in J$. Also, we know that $(t, v(t)) \in V$ for all $t \in I$, and hence it follows that $(t, u(t)) \in V$ for all $t \in J$ by the given property of V. So, the exact reasoning as in (‡) goes through, and hence this proves the claim.

8). In this problem, we will prove the Fundamental Estimate. Let $\Omega, v, L, \vec{\boldsymbol{\varphi}}, \vec{\boldsymbol{\psi}}$ and I be as in the problem statement.

As given in the problem, we define the following function on I.

$$u(t) = ||\vec{\boldsymbol{\varphi}}(t) - \vec{\boldsymbol{\psi}}(t)||^2$$

Also, define the function $\vec{\boldsymbol{v}}$ on I by the formula

$$v(t) = \left(\delta e^{L|t-t_0|} + \frac{\epsilon_1+\epsilon_2}{L}(e^{L|t-t_0|}-1)\right)^2$$

So to prove the claim, it is enough to show that

$$u(t) \leq v(t) \quad , \quad t \in I$$

To simplify our work, we will show this inequality in the interval $(t_0, \infty) \cap I$, and a similar argument will show that the inequality holds in the interval $(-\infty, t_0) \cap I$ as well. The condition $||\vec{\boldsymbol{\varphi}}(t_0) - \vec{\boldsymbol{\psi}}(t_0)||^2 \leq \delta^2$ shows that $u(t_0) \leq v(t_0)$.

First we show that

$$\dot{u} \leq 2Lu + 2\epsilon\sqrt{u} \tag{0.18}$$

Note that u can be written as the composition $||\cdot||^2 \circ (\vec{\boldsymbol{\varphi}} - \vec{\boldsymbol{\psi}})$. So by the chain rule, the derivative of u is given by

$$\dot{u}(t) = 2\langle \vec{\boldsymbol{\varphi}}(t) - \vec{\boldsymbol{\psi}}(t), \dot{\vec{\boldsymbol{\varphi}}}(t) - \dot{\vec{\boldsymbol{\psi}}}(t)\rangle \quad , \quad t \in I$$

where $\langle , \rangle$ is the usual inner product. So by the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\dot{u}(t) &\leq 2||\vec{\boldsymbol{\varphi}}(t) - \vec{\boldsymbol{\psi}}(t)|| \cdot ||\dot{\vec{\boldsymbol{\varphi}}}(t) - \dot{\vec{\boldsymbol{\psi}}}(t)|| \\
&= 2\sqrt{u(t)}||\dot{\vec{\boldsymbol{\varphi}}}(t) - v(t,\vec{\boldsymbol{\varphi}}(t)) + v(t,\vec{\boldsymbol{\varphi}}(t)) - v(t,\vec{\boldsymbol{\psi}}(t)) + v(t,\vec{\boldsymbol{\psi}}(t)) - \dot{\vec{\boldsymbol{\psi}}}(t)|| \\
&\leq 2\sqrt{u(t)}\left(||\dot{\vec{\boldsymbol{\varphi}}}(t) - v(t,\vec{\boldsymbol{\varphi}}(t))|| + ||v(t,\vec{\boldsymbol{\varphi}}(t)) - v(t,\vec{\boldsymbol{\psi}}(t))|| + ||\dot{\vec{\boldsymbol{\psi}}}(t) - v(t,\vec{\boldsymbol{\psi}}(t))||\right) \\
&\leq 2\sqrt{u(t)}\left(\epsilon_1 + L||\vec{\boldsymbol{\varphi}}(t) - \vec{\boldsymbol{\psi}}(t)|| + \epsilon_2\right) \\
&= 2\sqrt{u(t)}(\epsilon + L\sqrt{u(t)}) \\
&= 2Lu(t) + 2\epsilon\sqrt{u(t)}
\end{aligned}$$

and in the third last step, we have used the fact that $\vec{\boldsymbol{\varphi}}$ is an ϵ_1-approximation, $\vec{\boldsymbol{\psi}}$ is an ϵ_2-approximation and that $\vec{\boldsymbol{v}}$ is Lipschitz continuous. This proves inequality (0.18).

Next, define the function $f : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ by the formula

$$f(t,x) = 2Lx + 2\epsilon\sqrt{x}$$

where $\epsilon = \epsilon_1 + \epsilon_2$. Observe that for $x \in (0,\infty)$ we have

$$\frac{\partial f}{\partial x} = 2L + \frac{\epsilon}{\sqrt{x}}$$

Now, suppose $x \in [\delta^2, \infty)$. Then we have

$$0 < \frac{\partial f}{\partial x} \leq 2L + \frac{\epsilon}{\delta}$$

and hence $\partial f/\partial x$ is bounded for $x \in [\delta^2,\infty)$. So, it follows that f is Lipschitz continuous with respect to the second variable on $V = \mathbb{R} \times [\delta^2, \infty)$. Also, observe that V has the property mentioned in problem **7).** Clearly, we have that $v(t) \geq \delta^2$ for all $t \in I$, so that $(t, v(t)) \in V$ for each $t \in I$ and also $(t, u(t)) \in \mathbb{R} \times [0,\infty)$ for each $t \in I$ as well. Now the inequality (0.18) implies that

$$\dot{u}(t) \leq f(t, u(t)) \quad , \quad t \in I$$

Also for $t \in (t_0, \infty) \cap I$ we see that

$$\begin{aligned}
\dot{v}(t) &= 2\left(L\delta e^{L(t-t_0)} + \frac{\epsilon_1+\epsilon_2}{L}(Le^{L(t-t_0)})\right)\left(\delta e^{L(t-t_0)} + \frac{\epsilon_1+\epsilon_2}{L}(e^{L(t-t_0)}-1)\right) \\
&= 2L\left(\sqrt{v(t)} + \frac{\epsilon_1+\epsilon_2}{L}\right)\sqrt{v(t)} \\
&= 2Lv(t) + 2\epsilon\sqrt{v(t)} \\
&= f(t, v(t))
\end{aligned}$$

The inequality $u(t_0) \leq v(t_0)$ follows from the condition $||\vec{\boldsymbol{\varphi}}(t_0) - \vec{\boldsymbol{\psi}}(t_0)||^2 \leq \delta^2$. So, problem **7).** implies that

$$u(t) \leq v(t) \quad , \quad t \in (t_0, \infty) \cap I$$

As remarked above, we can prove the same inequality on $(t_0, \infty) \cap I$. This completes the proof of the claim.