HW-5

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1). Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & ... & 0 \\ & 0 & 1 & 0 & ... & 0 \\ & & 0 & 1 & ... & 0 \\ & & & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ a_0 & a_1 & a_2 & a_3 & ... & a_{n-1} \end{bmatrix}$$

i.e the main diagonal consists of only 0s, the super-diagonal consists of 1s and the bottom row is $a_0, ..., a_{n-1}$. We show that the characteristic polynomial of A upto sign is

$$t^n - a_{n-1}t^{n-1} - ... - a_1 t - a_0$$

and we will be using induction on n. To prove the base case, suppose $n = 2$. In that case, our matrix is of the form

$$A = \begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix}$$

and hence we have that

$$\text{char}(A) = \det(tI_2 - A) = t^2 - a_1 t - a_0$$

and hence the base case is true. So suppose the statement holds for some $n - 1 \in \mathbb{N}$, and we show it for n. We know that

$$\text{char}(A) = \det(tI_n - A)$$

and note that

$$tI_n - A = \begin{bmatrix} t\vec{e}_1 - \vec{e}_2 \\ t\vec{e}_2 - \vec{e}_3 \\ ... \\ t\vec{e}_{n-1} - \vec{e}_n \\ t\vec{e}_n - \vec{a} \end{bmatrix}$$

where $\vec{a} = (a_0, a_1, ..., a_{n-1})$. So, to calculate the determinant of $tI_n - A$, we expand along the first column. Put $\vec{a}' = (a_1, a_2, ..., a_{n-1})$. Note that the first column of the matrix $tI_n - A$ is the vector $(t, 0, ..., 0, -a_0)$. So expanding the determinant along this column, we get

$$\det(tI_n - A) = t \det \begin{bmatrix} t\vec{e}_1 - \vec{e}_2 \\ t\vec{e}_2 - \vec{e}_3 \\ ... \\ t\vec{e}_{n-2} - \vec{e}_{n-1} \\ t\vec{e}_{n-1} - \vec{a}' \end{bmatrix} + (-a_0)(-1)^{n-1}\det \begin{bmatrix} -1 & 0 & 0 & ... & 0 \\ t & -1 & 0 & ... & 0 \\ 0 & t & -1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & -1 \end{bmatrix}$$

Date: 11 February 2021.

In the second matrix above (which is an $(n-1)\times(n-1)$ matrix), we have all -1s in the main diagonal, and we have t directly below the main diagonal. For the first determinant, we can apply the induction hypothesis to get

$$t\det\begin{bmatrix} t\vec{\boldsymbol{e}}_1-\vec{\boldsymbol{e}}_2 \\ t\vec{\boldsymbol{e}}_2-\vec{\boldsymbol{e}}_3 \\ \dots \\ t\vec{\boldsymbol{e}}_{n-2}-\vec{\boldsymbol{e}}_{n-1} \\ t\vec{\boldsymbol{e}}_{n-1}-\vec{\boldsymbol{a}}' \end{bmatrix} = t(t^{n-1}-a_{n-1}t^{n-2}-\dots-a_2t-a_1)$$

The second determinant, being a lower triangular matrix, is simply the product of its diagonal elements, which is $(-1)^{n-1}$. So, it follows that

$$\det(tI_n-A)=t^n-a_{n-1}t^{n-1}-\dots-a_1t-a_0$$

and hence by induction, the claim has been proven.

2). This is just a straightforward computation. Suppose λ is an eigenvalue of A, and let $(v_1,v_2,\dots,v_n)$ be a corresponding eigenvector. So, we see that

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ & 0 & 1 & 0 & \dots & 0 \\ & & 0 & 1 & \dots & 0 \\ & & & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 & a_1 & a_2 & a_3 & \dots & a_{n-1} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \dots \\ \dots \\ v_{n-1} \\ v_n \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \dots \\ \dots \\ \lambda v_{n-1} \\ \lambda v_n \end{bmatrix}$$

Also, by the nature of the matrix A, we see that

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ & 0 & 1 & 0 & \dots & 0 \\ & & 0 & 1 & \dots & 0 \\ & & & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 & a_1 & a_2 & a_3 & \dots & a_{n-1} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \dots \\ \dots \\ v_{n-1} \\ v_n \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ \dots \\ \dots \\ v_n \\ a_0v_1+\dots+a_{n-1}v_n \end{bmatrix}$$

and this implies that

$$\begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \dots \\ \dots \\ \lambda v_{n-1} \\ \lambda v_n \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ \dots \\ \dots \\ v_n \\ a_0v_1+\dots+a_{n-1}v_n \end{bmatrix}$$

and this implies

$$v_i=\lambda^{i-1}v_1 \quad , \quad 1\le i\le n$$

and hence we see that

$$\begin{bmatrix} v_1 \\ v_2 \\ \dots \\ \dots \\ v_{n-1} \\ v_n \end{bmatrix} = v_1\begin{bmatrix} 1 \\ \lambda \\ \dots \\ \dots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} =$$

and hence this shows that the eigenspace corresponding to λ is one-dimensional and is spanned by the vector $(1,\lambda,\dots,\lambda^{n-1})$. Because the number of Jordan blocks associated to λ is equal to the geometric multiplicity of λ, this implies that there is precisely one Jordan block for each distinct eigenvalue of A. This proves the claim.

3). In this exercise, we compute the exponentials of the given matrices.

(a) $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

Solution. Since A is a diagonal matrix, we see that

$$A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

for every $n \in \mathbb{N}$. So, we see that

$$e^A = \sum_{m=0}^{\infty} \frac{A^m}{m!} = \sum_{m=0}^{\infty} \begin{bmatrix} \frac{a^m}{m!} & 0 \\ 0 & \frac{b^m}{m!} \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

and this is the required matrix. ■

(b) $A = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$, where $\theta \in \mathbb{R}$.

Solution. The first couple of powers of A are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}, \begin{bmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{bmatrix}, \begin{bmatrix} 0 & -\theta^3 \\ \theta^3 & 0 \end{bmatrix} \begin{bmatrix} \theta^4 & 0 \\ 0 & \theta^4 \end{bmatrix}, \begin{bmatrix} 0 & \theta^5 \\ -\theta^5 & 0 \end{bmatrix}, \dots$$

So, we see that

$$e^A = \sum_{m=0}^{\infty} \frac{A^m}{m!} = \begin{bmatrix} 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots & \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \\ -\theta + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \dots & 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

and this is the required exponential matrix. ■

4). Let $a, b \in \mathbb{R}$ and let

$$M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Let A be the $2n \times 2n$ real matrix

$$A = \begin{bmatrix} M & I_2 & & & 0 \\ & M & I_2 & & \\ & & M & \ddots & \\ & & & \ddots & I_2 \\ & & & & M \end{bmatrix}$$

We compute e^{tA}. Let P, Q be the matrices given by

$$P = \begin{bmatrix} M & & & & \\ & M & & & \\ & & M & & \\ & & & \ddots & \\ & & & & M \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & I_2 & & & \\ & 0 & I_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & I_2 \\ & & & & 0 \end{bmatrix}$$

i.e P is the matrix where in block form, all 2×2 diagonal matrices are all M, and Q is the matrix where in block form all the diagonal 2×2 matrices are 0s and the super-diagonal 2×2 matrices are all I_2. So, we see that

$$A = P + Q$$

Also, it is easy to see that $PQ = QP$, because

$$PQ = QP = \begin{bmatrix} 0 & M & & & \\ & 0 & M & & \\ & & 0 & \ddots & \\ & & & \ddots & M \\ & & & & 0 \end{bmatrix}$$

So, we see that

$$e^{tA} = e^{tP+tQ} = e^{tP}e^{tQ}$$

First, observe that

$$P^k = \begin{bmatrix} M^k & & & & \\ & M^k & & & \\ & & M^k & & \\ & & & \ddots & \\ & & & & M^k \end{bmatrix} \quad , \quad k \in \mathbb{N}$$

So, we see that

$$\begin{aligned} e^{tP} &= \sum_{m=0}^{\infty} \frac{t^m P^m}{m!} \\ &= \sum_{m=0}^{\infty} \begin{bmatrix} \frac{t^m M^m}{m!} & & & & \\ & \frac{t^m M^m}{m!} & & & \\ & & \frac{t^m M^m}{m!} & & \\ & & & \ddots & \\ & & & & \frac{t^m M^m}{m!} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{m=0}^{\infty} \frac{t^m M^m}{m!} & & & & \\ & \sum_{m=0}^{\infty} \frac{t^m M^m}{m!} & & & \\ & & \sum_{m=0}^{\infty} \frac{t^m M^m}{m!} & & \\ & & & \ddots & \\ & & & & \sum_{m=0}^{\infty} \frac{t^m M^m}{m!} \end{bmatrix} \\ &= \begin{bmatrix} e^{tM} & & & & \\ & e^{tM} & & & \\ & & e^{tM} & & \\ & & & \ddots & \\ & & & & e^{tM} \end{bmatrix} \end{aligned}$$

Now, we compute e^{tQ}. Before we do that, we introduce some notation. For each $1 \leq i \leq n$, let V_i be the $2 \times 2n$ matrix given by

$$V_i = \begin{bmatrix} 0 & 0 & ... & I_2 & ... & 0 \end{bmatrix}$$

where above, each 0 is the 2×2 zero matrix, and the i^{th} 2×2 block matrix above is I_2. Similarly, for $1 \leq i \leq n$ let W_i be the $2n \times 2$ matrix given by

$$W_i = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ I_2 \\ \ldots \\ 0 \end{bmatrix}$$

where above each 0 is the zero 2×2 matrix, and the i^{th} 2×2 block matrix is I_2. With this notation, we can multiply each V_i with W_j in the usual dot product way. Now, I claim that for each $1 \leq k < n$,

$$Q^k = \begin{bmatrix} V_{k+1} \\ V_{k+2} \\ \ldots \\ V_n \\ 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

where above, each 0 is the $2 \times 2n$ zero matrix. We prove the claim by induction on k. When $k = 1$, the claim is clear because

$$Q^1 = Q = \begin{bmatrix} V_2 \\ V_3 \\ \ldots \\ V_n \\ 0 \end{bmatrix}$$

Suppose the claim is true for some $1 \leq k < n - 1$, and we prove it for $k + 1$. Observe that we also have

$$Q = \begin{bmatrix} 0 & W_1 & W_2 & \ldots & W_{n-1} \end{bmatrix}$$

where above, the 0 is the $2n \times 2$ zero matrix. So, we see that

$$Q^{k+1} = Q^k Q = \begin{bmatrix} V_{k+1} \\ V_{k+2} \\ \ldots \\ V_n \\ 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & W_1 & W_2 & \ldots & W_{n-1} \end{bmatrix} = \begin{bmatrix} V_{k+2} \\ V_{k+3} \\ \ldots \\ V_n \\ 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

and this proves the claim. In particular, we have that

$$Q^{n-1} = \begin{bmatrix} V_n \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

where above, each 0 is the $2 \times 2n$ zero matrix. This means that $Q^n = 0$, i.e Q is a nilpotent matrix. So, we have

$$\begin{aligned} e^{tQ} &= \sum_{m=0}^{\infty} \frac{t^m Q^m}{m!} \\ &= \sum_{m=0}^{n-1} \frac{t^m Q^m}{m!} \\ &= I_{2n} + \frac{t}{1!}\begin{bmatrix} V_2 \\ V_3 \\ \dots \\ \dots \\ V_n \\ 0 \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} V_3 \\ V_4 \\ \dots \\ V_n \\ 0 \\ 0 \end{bmatrix} + \dots + \frac{t^{n-1}}{(n-1)!}\begin{bmatrix} V_n \\ 0 \\ \dots \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} I_2 & tI_2 & \frac{t^2}{2!}I_2 & \cdots & \frac{t^{(n-1)}}{(n-1)!}I_2 \\ & I_2 & tI_2 & \cdots & \frac{t^{(n-2)}}{(n-2)!}I_2 \\ & & I_2 & \cdots & \frac{t^{(n-3)}}{(n-3)!}I_2 \\ & & & \ddots & \\ & & & \cdots & tI_2 \\ & & & \cdots & I_2 \end{bmatrix} \end{aligned}$$

where the last matrix is in block-matrix form. So, it follows that

$$\begin{aligned} e^{tA} &= e^{tP}e^{tQ} \\ &= \begin{bmatrix} e^{tM} & & & & & \\ & e^{tM} & & & & \\ & & e^{tM} & & & \\ & & & e^{tM} & & \\ & & & & \ddots & \\ & & & & & e^{tM} \end{bmatrix} \begin{bmatrix} I_2 & tI_2 & \frac{t^2}{2!}I_2 & \cdots & \frac{t^{(n-1)}}{(n-1)!}I_2 \\ & I_2 & tI_2 & \cdots & \frac{t^{(n-2)}}{(n-2)!}I_2 \\ & & I_2 & \cdots & \frac{t^{(n-3)}}{(n-3)!}I_2 \\ & & & \ddots & \\ & & & \cdots & tI_2 \\ & & & \cdots & I_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{tM} & te^{tM} & \frac{t^2}{2!}e^{tM} & \cdots & \frac{t^{(n-1)}}{(n-1)!}e^{tM} \\ & e^{tM} & te^{tM} & \cdots & \frac{t^{(n-2)}}{(n-2)!}e^{tM} \\ & & e^{tM} & \cdots & \frac{t^{(n-3)}}{(n-3)!}e^{tM} \\ & & & \ddots & \\ & & & \cdots & te^{tM} \\ & & & \cdots & e^{tM} \end{bmatrix} \end{aligned}$$

and so our required matrix B is

$$B = e^{tM}$$

Finally, we compute the matrix B. Observe that

$$M = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = aI_2 + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = E + F$$

and clearly the two matrices E, F in the above sum commute. So, it follows that

$$e^{tM} = e^{tE}e^{tF}$$

Now, observe that

$$tE = taI_2 = \begin{bmatrix} ta & 0 \\ 0 & ta \end{bmatrix}$$

So by part (a) of problem **3).** we see that

$$e^{tE} = \begin{bmatrix} e^{ta} & 0 \\ 0 & e^{ta} \end{bmatrix}$$

Also, observe that

$$tF = \begin{bmatrix} 0 & -tb \\ tb & 0 \end{bmatrix}$$

and so by (b) of problem **3).** we see that

$$e^{tF} = \begin{bmatrix} \cos(tb) & -\sin(tb) \\ \sin(tb) & \cos(tb) \end{bmatrix}$$

and combining all of this, we see that

$$\begin{aligned} e^{tM} &= e^{tE}e^{tF} \\ &= \begin{bmatrix} e^{ta} & 0 \\ 0 & e^{ta} \end{bmatrix} \begin{bmatrix} \cos(tb) & -\sin(tb) \\ \sin(tb) & \cos(tb) \end{bmatrix} \\ &= \begin{bmatrix} e^{ta}\cos(tb) & -e^{ta}\sin(tb) \\ e^{ta}\sin(tb) & e^{ta}\cos(tb) \end{bmatrix} \end{aligned}$$

and this is the required matrix B.

5). Define the one-parameter group of diffeomorhpisms $\{g^t\}$ on $M = (0,1)$ by the following formula.

$$g^t x = \frac{xe^t}{xe^t + 1 - x} \quad , \quad t \in \mathbb{R}, x \in M$$

It is clear that for $x \in M$ and $t \in \mathbb{R}$, $g^t x \in M$, because the numerator is always strictly smaller than the denominator. Also, g is $\mathscr{C}^2$ on $\mathbb{R} \times M$, because it has continuous partial derivatives. Let us first show that this is indeed a one-parameter group. So let $s, t \in \mathbb{R}$. So we have

$$\begin{aligned} g^s(g^t x) &= g^s\left(\frac{xe^t}{xe^t + 1 - x}\right) \\ &= \frac{\frac{xe^t e^s}{xe^t + 1 - x}}{\frac{xe^t e^s}{xe^t + 1 - x} + 1 - \frac{xe^t}{xe^t + 1 - x}} \\ &= \frac{xe^t e^s}{xe^t e^s + xe^t + 1 - x - xe^t} \\ &= \frac{xe^{s+t}}{xe^{s+t} + 1 - x} \\ &= g^{s+t}x \end{aligned}$$

and hence this shows that

$$g^s g^t = g^{s+t}$$

Now this condition automatically forces each g^t to be a diffeomorphism from M to itself. So, $\{g^t\}$ is indeed a one-parameter group.

Next, we compute the velocity vector field v. For any $x \in M$, we have

$$\begin{aligned} v(x) &= \lim_{h\to 0} \frac{g^h x - x}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{xe^h}{xe^h + 1 - x} - x}{h} \\ &= \lim_{h\to 0} \frac{xe^h - x^2e^h - x + x^2}{(xe^h + 1 - x)h} \\ &= \lim_{h\to 0} \frac{x(e^h - 1) - x^2(e^h - 1)}{h(xe^h + 1 - x)} \\ &= \lim_{h\to 0} \frac{e^h - 1}{h}\left(\frac{x - x^2}{xe^h + 1 - x}\right) \\ &= x - x^2 \\ &= x(1 - x) \end{aligned}$$

So, the corresponding autonomous differential equation for this one-parameter group is

$$\dot{x} = v(x) = x(1 - x)$$

and hence this is the required one-parameter group.

6). Consider the maps

$$g^t x = \frac{x}{x + (1 - x)e^t} \quad , \quad x \in M, t \in \mathbb{R}$$

If $x \in M$ and $t \in \mathbb{R}$, then the numerator is strictly smaller than the denominator, and hence $g^t x \in M$. Moreover, g is $\mathscr{C}^2$ on $\mathbb{R} \times M$ becaus it has continuous partial derivatives. Now, let $s, t \in \mathbb{R}$. Then we have the following chain of equations.

$$\begin{aligned} g^s(g^t x) &= g^s\left(\frac{x}{x + (1 - x)e^t}\right) \\ &= \frac{x}{x + (x + (1 - x)e^t - x)e^s} \\ &= \frac{x}{x + (1 - x)e^t e^s} \\ &= \frac{x}{x + (1 - x)e^{s+t}} \\ &= g^{s+t} x \end{aligned}$$

and hence this implies that

$$g^{s+t} = g^s g^t$$

showing that $\{g^t\}$ is indeed a one-parameter group of diffeomorphisms on M (the above condition enforces every g^t to be a diffeomorphism).

Next, we compute the phase velocity field of this parameter group. For $x \in M$, we have

$$\begin{aligned} v(x) &= \lim_{h\to 0}\frac{g^h x - x}{h} \\ &= \lim_{h\to 0}\frac{\dfrac{x}{x+(1-x)e^h} - x}{h} \\ &= \lim_{h\to 0}\frac{x - x^2 - x(1-x)e^h}{(x+(1-x)e^h)h} \\ &= \lim_{h\to 0}\frac{x(1-x)(1-e^h)}{h(x+(1-x)e^h} \\ &= \lim_{h\to 0} -\left(\frac{e^h-1}{h}\right)\frac{x(1-x)}{x+(1-x)e^h} \\ &= x(x-1) \end{aligned}$$

and so the corresponding autonomous differential equation is

$$\dot{x} = x(x-1)$$

7). In this problem, we will sketch the intergral curves in the extended phase space of the one-parameter groups occuring in the previous two problems. Observe that the extended phase space in both the problems is $\mathbb{R} \times (0,1)$.

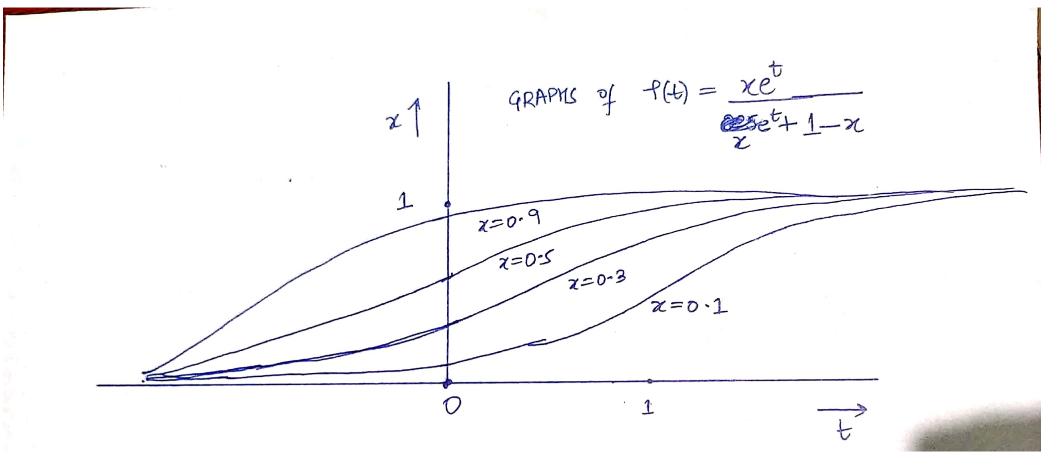
Integral Curves for problem 5). The one-parameter group is

$$g^t x = \frac{xe^t}{xe^t + 1 - x} \quad , \quad t \in \mathbb{R}, x \in M$$

For a fixed $x \in (0,1)$, we sketch the graph of the function

$$\varphi(t) = \frac{xe^t}{xe^t + 1 - x} \quad , \quad t \in \mathbb{R}$$

The graphs are given in the picture below.



Integral Curves for problem 6). The one-parameter group is

$$g^t x = \frac{x}{x + (1-x)e^t} \quad , \quad t \in \mathbb{R}, x \in M$$

For different values of $x \in (0,1)$, we sketch the graph of the function

$$\varphi(t) = \frac{x}{x + (1-x)e^t} \quad , \quad t \in \mathbb{R}$$

The graphs are given in the picture below.