HW-6

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1). In this problem, we do problems **8** and **18** from Cookbook-II.

8 $\quad x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + (x^2-1)\dfrac{\mathrm{d}y}{\mathrm{d}x} + x^3y = 0, \quad x > 0$

Solution. Dividing throughout by x, we get the equation

$$\frac{\mathrm{d}y^2}{\mathrm{d}x^2} + \frac{(x^2-1)}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + x^2y = 0$$

Put

$$p(x) = \frac{(x^2-1)}{x} \quad , \quad q(x) = x^2$$

Observe that

$$\frac{q'(x) + 2p(x)q(x)}{2(q(x))^{3/2}} = \frac{2x + 2x(x^2-1)}{2x^3} = 1$$

i.e this quantity is constant on the interval of existence. So, consider the transformation

$$t = \int \sqrt{q(x)}\,\mathrm{d}x = \int x\,\mathrm{d}x = \frac{x^2}{2}$$

For these transformations we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}x^2}$$

In our case, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{2t}\frac{\mathrm{d}y}{\mathrm{d}t} \quad , \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t}$$

and so the equation becomes

$$2t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2t-1}{\sqrt{2t}}\sqrt{2t}\frac{\mathrm{d}y}{\mathrm{d}t} + 2ty = 0$$

which is the same as

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + y = 0$$

Note this this is a DE with constant coefficients, and this can be easily solved. The characteristic equation is

$$s^2 + s + 1 = 0$$

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The roots of this polynomial are $\frac{-1\pm i\sqrt{3}}{2}$, and each root has multiplicity 1. So the general solution of the DE is given by

$$y(t) = c_1 e^{\frac{-t}{2}} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{\frac{-t}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

where $c_1, c_2 \in \mathbb{R}$ are constants. So, in terms of x, the general solution is

$$y = c_1 e^{\frac{-x^2}{4}} \cos\left(\frac{\sqrt{3}}{4}x^2\right) + c_2 e^{\frac{-x^2}{4}} \sin\left(\frac{\sqrt{3}}{4}x^2\right)$$

where $c_1, c_2 \in \mathbb{R}$. ■

18 $y^{(6)} - 3y^{(5)} + 45y^{(4)} - 24y^{(3)} + 236y'' + 1300y' - 4056y = x^2 e^x \cos(\sqrt{5}x)$

Solution. First, we write the given DE in the form

$$y^{(6)} - 3y^{(5)} + 45y^{(4)} - 24y^{(3)} + 236y'' + 1300y' - 4056y = e^{1\cdot x}(x^2 \cdot \cos(\sqrt{5}x) + 0 \cdot \sin(\sqrt{5}x))$$

and hence $\alpha = 1, \beta = \sqrt{5}$ where α, β are as in Cookbook-II. So, the *form* of a particular solution of this DE will be

$$y_p = x^s e^x \left(A(x)\cos(\sqrt{5}x) + B(x)\sin(\sqrt{5}x)\right)$$

where A, B are polynomials of degree 2 and s is the multiplicity of $r = 1 + i\sqrt{5}$ as a root of the characteristic polynomial. Let us calculate this multiplicity. The characteristic polynomial is

$$t^6 - 3t^5 + 45t^4 - 24t^3 + 236t^2 + 1300t - 4056$$

If $1 + i\sqrt{5}$ is a root, then $1 - i\sqrt{5}$ is also a root, and hence $(x-1)^2 + 5 = x^2 - 2x + 6$ must be a factor of this polynomial. But this is not true, and hence the multiplicity $s = 0$. ■

2). Let Ω be a domain in $\mathbb{R} \times \mathbb{R}^n$ and let $\vec{\boldsymbol{v}} : \Omega \to \mathbb{R}^n$ be a $\mathscr{C}^1$ function. We show that $\vec{\boldsymbol{v}}$ is locally Lipschitz in the second argument.

Let $(t_0, \vec{\boldsymbol{a}}_0) \in \Omega$. First, choose some $r > 0$ such that

$$\overline{B}((t_0, \vec{\boldsymbol{a}}_0), r) \subseteq \Omega$$

and this is possible because Ω is open. Now, we know that $\vec{\boldsymbol{v}}$ is a $\mathscr{C}^1$ function; hence, it follows that the map $\vec{\boldsymbol{v}}' : \overline{B}((t_0, \vec{\boldsymbol{a}}_0), r) \to \mathbb{R}^{n(n+1)}$ given by $(t, \vec{\boldsymbol{a}}) \mapsto \vec{\boldsymbol{v}}'(t, \vec{\boldsymbol{a}})$ is a continuous map. If $||\cdot||_\circ$ denotes the operator norm, this means that the map $(t, \vec{\boldsymbol{a}}) \mapsto ||\vec{\boldsymbol{v}}'(t, \vec{\boldsymbol{a}})||_\circ$ is continuous, where we are interpreting $\vec{\boldsymbol{v}}'(t, \vec{\boldsymbol{a}})$ as an $n \times (n+1)$ matrix. So, there is some $M > 0$ such that

$$||\vec{\boldsymbol{v}}'(t, \vec{\boldsymbol{a}})||_\circ \leq M$$

for all $(t, \vec{\boldsymbol{a}}) \in \overline{B}((t_0, \vec{\boldsymbol{a}}_0), r)$, and this is true because $\overline{B}((t_0, \vec{\boldsymbol{a}}_0), r)$ is a compact set.

Now, suppose $(t, \vec{\boldsymbol{a}}_1)$ and $(t, \vec{\boldsymbol{a}}_2)$ are any two points in $\overline{B}((t_0, \vec{\boldsymbol{a}}_0), r)$. We know that $\overline{B}((t_0, \vec{\boldsymbol{a}}_0), r)$ is a convex set; so, we can define $\vec{\boldsymbol{g}} : [0,1] \to \overline{B}((t_0, \vec{\boldsymbol{a}}_0), r)$ to be the linear path between these two points, i.e

$$\vec{\boldsymbol{g}}(s) = (1-s)(t, \vec{\boldsymbol{a}}_1) + s(t, \vec{\boldsymbol{a}}_2) \quad , \quad t \in [0,1]$$

Next, consider the map $\vec{\boldsymbol{v}} \circ \vec{\boldsymbol{g}} : [0,1] \to \mathbb{R}^n$, which is clearly a $\mathscr{C}^1$ map because both $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{g}}$ are $\mathscr{C}^1$. Also, observe that for any $s \in [0,1]$ we have by the chain rule

$$(\vec{\boldsymbol{v}} \circ \vec{\boldsymbol{g}})'(s) = \vec{\boldsymbol{v}}'(\vec{\boldsymbol{g}}(s))\vec{\boldsymbol{g}}'(s) = \vec{\boldsymbol{v}}'(\vec{\boldsymbol{g}}(s))(0, \vec{\boldsymbol{a}}_2 - \vec{\boldsymbol{a}}_1)$$

where in the above equation, we are interpretting $\vec{\boldsymbol{v}}'(\vec{\boldsymbol{g}}(s))$ as a matrix. So, we see that

$$||(\vec{\boldsymbol{v}} \circ \vec{\boldsymbol{g}})'(s)|| = ||\vec{\boldsymbol{v}}'(\vec{\boldsymbol{g}}(s))||_{\circ}||(0, \vec{\boldsymbol{a_2}} - \vec{\boldsymbol{a_1}})|| \leq M||\vec{\boldsymbol{a_2}} - \vec{\boldsymbol{a}}_1|| \quad , \quad s \in [0,1]$$

So, by the mean-value theorem in $\mathbb{R}^n$, we immediately see that

$$\begin{aligned} ||\vec{\boldsymbol{v}}(t, \vec{\boldsymbol{a}}_2) - \vec{\boldsymbol{v}}(t, \vec{\boldsymbol{a}}_1)|| &= ||\vec{\boldsymbol{v}} \circ \vec{\boldsymbol{g}}(1) - \vec{\boldsymbol{v}} \circ \vec{\boldsymbol{g}}(0)|| \\ &\leq M||\vec{\boldsymbol{a}}_2 - \vec{\boldsymbol{a}}_1||(1-0) \\ &=\leq M||\vec{\boldsymbol{a}}_2 - \vec{\boldsymbol{a}}_1|| \end{aligned}$$

and this shows that $\vec{\boldsymbol{v}}$ is locally Lipschitz in the second argument.

3). Suppose $J_{\vec{\boldsymbol{a}}} = \mathbb{R}$ for all $\vec{\boldsymbol{a}} \in U$. We show that $\{g^t\}$ is a one-parameter group of transformations on U, where

$$g^t(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{\varphi}}_{\vec{\boldsymbol{x}}}(t) \quad , \quad \vec{\boldsymbol{x}} \in U, t \in \mathbb{R}$$

We only need to show that for all $s, t \in \mathbb{R}$, we have

$$g^{s+t} = g^s g^t \tag{0.1}$$

To prove this, let $s, t \in \mathbb{R}$ and take any $\vec{\boldsymbol{x}} \in U$. Consider the maps $\vec{\boldsymbol{\varphi}}_{\vec{\boldsymbol{x}}}$ and $\vec{\boldsymbol{\varphi}}_{g^t(\vec{\boldsymbol{x}})}$. We will show that

$$\vec{\boldsymbol{\varphi}}_{g^t(\vec{\boldsymbol{x}})}(u) = \vec{\boldsymbol{\varphi}}_{\vec{\boldsymbol{x}}}(u+t) \quad , \quad u \in \mathbb{R} \tag{0.2}$$

To show this, first observe that

$$\dot{\vec{\boldsymbol{\varphi}}}_{\vec{\boldsymbol{x}}} = \vec{\boldsymbol{v}}(\vec{\boldsymbol{\varphi}}_{\vec{\boldsymbol{x}}}) \quad , \quad \vec{\boldsymbol{\varphi}}_{\vec{\boldsymbol{x}}}(0) = \vec{\boldsymbol{x}}$$

and also $\vec{\boldsymbol{\varphi}}_{\vec{\boldsymbol{x}}}$ is the unique solution to the above IVP. Now put $\vec{\boldsymbol{\psi}}(u) = \vec{\boldsymbol{\varphi}}_{\vec{\boldsymbol{x}}}(u+t)$ for $u \in \mathbb{R}$. From the above equations, it is clear that

$$\dot{\vec{\boldsymbol{\psi}}} = \vec{\boldsymbol{v}}(\vec{\boldsymbol{\psi}}) \quad , \quad \vec{\boldsymbol{\psi}}(0) = \vec{\boldsymbol{\varphi}}_{\vec{\boldsymbol{x}}}(t) = g^t(\vec{\boldsymbol{x}})$$

But by definition, we know that the map $\vec{\boldsymbol{\varphi}}_{g^t(\vec{\boldsymbol{x}})}$ is a solution to the above IVP as well. By uniqueness, it follows that $\vec{\boldsymbol{\psi}} = \vec{\boldsymbol{\varphi}}_{g^t(\vec{\boldsymbol{x}})}$, and this proves equation (0.2). This implies that

$$\begin{aligned} g^{s+t}(\vec{\boldsymbol{x}}) &= \vec{\boldsymbol{\varphi}}_{\vec{\boldsymbol{x}}}(s+t) \\ &= \vec{\boldsymbol{\varphi}}_{g^t(\vec{\boldsymbol{x}})}(s) \\ &= g^s(g^t(\vec{\boldsymbol{x}})) \end{aligned}$$

and this proves equation (0.1), and hence shows that $\{g^t\}$ is a one-parameter group of transformations on U.

4). Suppose $\vec{\boldsymbol{v}}$ is $\mathscr{C}^1$, and that $J_a = \mathbb{R}$ for all $a \in U$. We show that $\{g^t\}$ is a one-parameter group of diffeomorphisms. We have already shown that this is a one-parameter group of transformations. So, we only need to show that the map $g : \mathbb{R} \times U \to \mathbb{R}^n$ defined by $g(t, \vec{\boldsymbol{x}}) = g^t\vec{\boldsymbol{x}}$ is a $\mathscr{C}^1$ map on U. But, this clearly is a consequence of **Theorem 1** as mentioned in the homework sheet.

5). Let $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We show that there is a one parameter group $\{g^t\}$ of diffeomorphisms on I whose phase velocity field is given by $v(x) = \cos x$.

We know that if $\{g^t\}$ is such a one-parameter group, then for any $x_0 \in I$ the map $\varphi(t) = g^t x_0$ is the unique solution to the IVP

$$\dot{y}(t) = \cos(y(t)) \quad , \quad y(0) = x_0$$

So, the idea is to solve the DE

$$\dot{y}(t) = \cos(y(t))$$

which is equivalent to the DE

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \cos y$$

and clearly this is a separable DE. We get

$$\sec y \frac{\mathrm{d}y}{\mathrm{d}t} = 1$$

Integrating both sides with respect to t, we get

$$\int \sec y \,\mathrm{d}y = t + C$$

for some $C \in \mathbb{R}$ and hence

$$\log(\tan y + \sec y) = t + C$$

for some $C \in \mathbb{R}$. By putting $t = 0$, the value of C obtained is

$$C = \log(\tan x_0 + \sec x_0)$$

This gives us

$$\tan y + \sec y = Ke^t$$

where $K = \tan x_0 + \sec x_0$. Now, observe that $y(t) \in I$ for all $I \in \mathbb{R}$, and hence $\sec y(t) \geq 0$ for all $t \in \mathbb{R}$. This means that

$$\sec y = \sqrt{1 + \tan^2 y}$$

and hence

$$\tan y + \sqrt{1 + \tan^2 y} = Ke^t$$

This gives us

$$1 + \tan^2 y = K^2 e^{2t} + \tan^2 y - 2Ke^t \tan y$$

and hence

$$\tan y = \frac{K^2 e^{2t} - 1}{2Ke^t}$$

and hence

$$y = \arctan\left(\frac{K^2 e^{2t} - 1}{2Ke^t}\right)$$

So, we may define

$$g^t x = \arctan\left(\frac{(\tan x + \sec x)^2 e^{2t} - 1}{2(\tan x + \sec x)e^t}\right)$$

6). We just use the results of problem 5) for this problem. Let $x_0 = 0$. So, as in the previous problem, we get $K = 1$, and hence the map

$$y(t) = \arctan\left(\frac{e^{2t} - 1}{2e^t}\right)$$

satisfies

$$\dot{y}(t) = \cos(y(t)) \quad , \quad y(0) = 0$$

for all $t \in \mathbb{R}$. So, y is a map from $(-\infty, \infty) \to I$, where I is the interval as in the previous problem. Now, by reversing the steps in the previous problem, we can see get that for all $t \in \mathbb{R}$,

$$\log(\tan(y(t)) + \sec(y(t))) = t$$

(here $C = \log K = 0$). So, this means that the inverse map $\theta : I \to (-\infty, \infty)$ is given by

$$\theta(x) = \log(\tan x + \sec x)$$

By the results in section 1.4.5 of the main notes, we see that

$$\lim_{x \to -\frac{\pi}{2}^+} \theta(x) = \lim_{x \to -\frac{\pi}{2}^+} \log(\tan x + \sec x) = -\infty$$

Exponentiating both sides, we see that

$$\lim_{x \to -\frac{\pi}{2}^+} \tan x + \sec x = e^{-\infty} = 0$$

and this completes the proof of the claim.