HW-7

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1). Here, I will be solving problems **19** and **20** from Cookbook-II.

19 Here, we show that the transformation $t = \ln x$ transforms the DE in equation (5) of the text-portion of Cookbook-II to the one in (6).

Solution. Suppose we have a DE of the form

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \alpha x \frac{\mathrm{d}y}{\mathrm{d}x} + \beta y = 0, \quad x > 0$$

where α, β are constants. Let us substitute $t = \ln x$. By the chain rule, we first have the following equations.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2 t}{\mathrm{d}x^2}$$

In our case, we have

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{x} = \frac{1}{e^t} \quad , \quad \frac{\mathrm{d}^2 t}{\mathrm{d}x^2} = -\frac{1}{x^2} = -\frac{1}{e^{2t}}$$

So using the above two equations, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{e^t}\frac{\mathrm{d}y}{\mathrm{d}t}$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\frac{1}{e^{2t}} + \frac{\mathrm{d}y}{\mathrm{d}t}\left(-\frac{1}{e^{2t}}\right)$$

So, the original DE becomes

$$e^{2t}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\frac{1}{e^{2t}} + \frac{\mathrm{d}y}{\mathrm{d}t}\left(-\frac{1}{e^{2t}}\right)\right) + \alpha e^t\left(\frac{1}{e^t}\frac{\mathrm{d}y}{\mathrm{d}t}\right) + \beta y = 0$$

which gives us

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} + \alpha\frac{\mathrm{d}y}{\mathrm{d}t} + \beta y = 0$$

which is the same as the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (\alpha - 1)\frac{\mathrm{d}y}{\mathrm{d}t} + \beta y = 0$$

and this finishes the proof. ■

20 Here we show that the transformation $t = \int q(x)^{\frac{1}{2}}\,\mathrm{d}x$ transforms the DE in (7) in the text portion of Cookbook-II to a linear DE with constant co-efficients.

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Solution. Suppose we have an equation of the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0$$

on an interval I such that $q > 0$ on I and

$$\frac{q'(x) + 2p(x)q(x)}{2(q(x))^{3/2}} = c$$

for some $c \in \mathbf{R}$ on I. Consider the transformation

$$t = \int \sqrt{q(x)}\,\mathrm{d}x$$

Again, by the chain rule, we first have the following equations.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}x^2}$$

In our case, we have

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \sqrt{q(x)} \quad , \quad \frac{\mathrm{d}^2t}{\mathrm{d}x^2} = \frac{q'(x)}{2\sqrt{q(x)}}$$

Putting all this together, we have the following.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\sqrt{q(x)}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\left(\sqrt{q(x)}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{q'(x)}{2\sqrt{q(x)}}$$

So, our original equation becomes

$$q(x)\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{q'(x)}{2\sqrt{q(x)}}\frac{\mathrm{d}y}{\mathrm{d}t} + p(x)\sqrt{q(x)}\frac{\mathrm{d}y}{\mathrm{d}t} + q(x)y = 0$$

which is the same as the equation

$$q(x)\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{q'(x) + 2p(x)q(x)}{2\sqrt{q(x)}}\frac{\mathrm{d}y}{\mathrm{d}t} + q(x)y = 0$$

Dividing throughout by $q(x)$, we get

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{q'(x) + 2p(x)q(x)}{2q(x)^{3/2}}\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0$$

which by assumption is the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + c\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0$$

which is clearly a linear DE with constant coefficients. This proves the claim. ■

2). Let $\boldsymbol{v}$ be the vector field on $\Omega = \{(x, y, z) \mid y \neq 0\}$ given by

$$\boldsymbol{v} = ((y^2 + z^2)y^{-1}, xz, -xy)$$

We first find two first integrals for $\boldsymbol{v}$ on a suitable large open subset U of Ω. Note that the corresponding DE is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (y^2 + z^2)y^{-1} \\ xz \\ -xy \end{bmatrix}$$

Let U be the subset of Ω given by $U = \{(x, y, z) \mid x \neq 0, y \neq 0\}$. On U, observe that

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{xz}{-xy} = \frac{-z}{y}$$

and this is clearly a separable DE. Solving this, we get

$$\int y \,\mathrm{d}y = \int -z \,\mathrm{d}z$$

and this gives us

$$\frac{y^2}{2} = \frac{-z^2}{2} + c_1'$$

for some $c_1' \in \mathbf{R}$. Rearranging, we get

$$y^2 + z^2 = 2c_1' = c_1$$

where $c_1 \in \mathbf{R}$. So, if we define the function $f : U \to \mathbf{R}$ given by

$$f(x, y, z) = y^2 + z^2$$

then it follows that f is a first integral for $\boldsymbol{v}$. This can be easily verified by checking the equation

$$(y^2 + z^2)y^{-1}\frac{\partial f}{\partial x} + xz\frac{\partial f}{\partial y} - xy\frac{\partial f}{\partial z} \equiv 0$$

on U. So, one first integral has been found.

Now, using the fact that $y^2 + z^2 = c_1$ on U (and hence $c_1 > 0$), we see that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c_1}{y}$$

on U. So, we get

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{c_1}{y}\frac{1}{-xy} = \frac{-c_1}{xy^2} = \frac{-c_1}{x(c_1 - z^2)}$$

and again we have a separable DE. Upon rearranging, we get

$$x\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{-c_1}{(c_1 - z^2)}$$

and hence

$$\int x \,\mathrm{d}x = -c_1 \int \frac{1}{c_1 - z^2} \,\mathrm{d}z$$

which gives us

$$\frac{x^2}{2} = -c_1\left(\frac{\log\left(\frac{z}{\sqrt{c_1}} + 1\right)}{2\sqrt{c_1}} - \frac{\log\left(1 - \frac{z}{\sqrt{c_1}}\right)}{2\sqrt{c_1}} + c_2'\right)$$

for some $c_2' \in \mathbf{R}$, and this can be written as

$$\frac{x^2}{2} = \frac{\sqrt{c_1}}{2}\left(\log\left(1 - \frac{z}{\sqrt{c_1}}\right) - \log\left(1 + \frac{z}{\sqrt{c_1}}\right)\right) + c_2$$

where $c_2 = -c_1 c_2' \in \mathbf{R}$, and hence

$$\frac{x^2}{2} = \frac{\sqrt{c_1}}{2}\log\left(\frac{\sqrt{c_1} - z}{\sqrt{c_1} + z}\right) + c_2$$

Now, replacing c_1 by $y^2 + z^2$ above, we get

$$\frac{x^2}{2} = \frac{\sqrt{y^2+z^2}}{2}\log\left(\frac{\sqrt{y^2+z^2} - z}{\sqrt{y^2+z^2} + z}\right) + c_2$$

and this gives us

$$\frac{x^2}{2} - \frac{\sqrt{y^2+z^2}}{2}\log\left(\frac{\sqrt{y^2+z^2} - z}{\sqrt{y^2+z^2} + z}\right) = c_2$$

So if we define the function $g : U \to \mathbf{R}$ by

$$g(x,y,z) = \frac{x^2}{2} - \frac{\sqrt{y^2+z^2}}{2}\log\left(\frac{\sqrt{y^2+z^2} - z}{\sqrt{y^2+z^2} + z}\right)$$

then it follows that g is a first integral for v.

Now, we show that the level surfaces $f = c_1$ and $g = c_2$ intersect transversally. To show this, it is enough to show that the Jacobian matrix $\partial(f,g)/\partial(x,y,z)$ has rank 2 at every point of U. Now, we have

$$\frac{\partial(f,g)}{\partial(x,y,z)} = \begin{bmatrix} 0 & 2y & 2z \\ x & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \end{bmatrix}$$

Observe that on U, we have $x \neq 0$. So, it follows that the rank of the Jacobian is always 2 (because the two rows cannot be scalar multiplies of each other), and this proves the claim.

3). Let $\boldsymbol{v}$ be the vector field on $\mathbf{R}^3$ given by

$$\boldsymbol{v} = \begin{bmatrix} y+z \\ y \\ x-y \end{bmatrix}$$

Let U be the open subset of $\mathbf{R}^3$ on which $z^2 > (x-y)^2$ and $y > 0$.

(a) First we find two first integrals f and g for $\boldsymbol{v}$ on U such that their level surfaces intersect transversally. The corresponding DE is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y+z \\ y \\ x-y \end{bmatrix}$$

From here, we can see that

$$\frac{\mathrm{d}(x+z)}{dt} = x + z$$

and hence we see that

$$\frac{\mathrm{d}(x+z)}{\mathrm{d}y} = \frac{x+z}{y}$$

This implies that

$$\frac{1}{y^2}\left(y\frac{\mathrm{d}(x+z)}{\mathrm{d}y}-(x+z)\right)=0$$

which is the same as writing

$$\frac{1}{y}\frac{\mathrm{d}(x+z)}{\mathrm{d}y}-\frac{1}{y^2}(x+z)=\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{y}(x+z)\right)=0$$

and hence this implies that

$$\frac{x+z}{y}=c_1$$

for some $c_1 \in \mathbf{R}$. So, if we define the function $f : U \to \mathbf{R}$ given by

$$f(x,y,z)=\frac{x+z}{y}$$

then it follows that f is a first integral for $\boldsymbol{v}$ on U, and this easily follows by verifying the equation

$$(y+z)\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+(x-y)\frac{\partial f}{\partial z}\equiv 0 \qquad \text{on } U$$

Also from the given DE we can see that

$$\frac{\mathrm{d}(x-y)}{\mathrm{d}t}=z$$

and hence we see that

$$\frac{\mathrm{d}(x-y)}{\mathrm{d}z}=\frac{z}{x-y}$$

This is again a separable DE which we can solve. By integrating we get

$$\int (x-y)\,\mathrm{d}(x-y)=\int z\,\mathrm{d}z=\frac{z^2}{2}+c_2'$$

for some $c_2' \in \mathbf{R}$. This gives us

$$(x-y)^2-z^2=2c_2'=c_2$$

for all $(x,y,z) \in U$. So if we define a function $g : U \to \mathbf{R}$ given by

$$g(x,y,z)=(x-y)^2-z^2$$

then it follows that g is a first integral for $\boldsymbol{v}$ on U. Again, this can be easily verified by checking the equation

$$(y+z)\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}+(x-y)\frac{\partial g}{\partial z}\equiv 0 \qquad \text{on } U$$

So, we have found two first integrals f, g on U. We now show that the level surfaces $f = c_1$ and $g = c_2$ intersect transversally, and to show this we show that $\partial(f,g)/\partial(x,y,z)$ has rank 2 on U. Observe that

$$\frac{\partial(f,g)}{\partial(x,y,z)}=\begin{bmatrix}\frac{1}{y} & \frac{-(x+z)}{y^2} & \frac{1}{y}\\ 2(x-y) & -2(x-y) & -2z\end{bmatrix}$$

Now, I claim that this matrix must have rank 2. Because $y > 0$ on U, it is clear that this matrix has rank atleast 1. For the sake of contradiction, suppose the matrix has rank 2. So, there is some non-zero $\lambda \in \mathbf{R}$ such that

$$\begin{bmatrix} \frac{1}{y} \\ \frac{-(x+z)}{y^2} \\ \frac{1}{y} \end{bmatrix} = \begin{bmatrix} 2\lambda(x-y) \\ -2\lambda(x-y) \\ -2\lambda z \end{bmatrix}$$

This implies that

$$2\lambda(x-y) = -2\lambda z$$

and this implies that

$$(x-y)^2 = z^2$$

which is a contradiction, since $(x, y, z) \in U$. So, it follows that the Jacobian has rank 2 at all points of U, and hence the level surfaces intersect transversally.

(b) Let (x_0, y_0, z_0) be a point in U. We will find a solution to the IVP

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}) \quad , \quad \boldsymbol{x}(0) = (x_0, y_0, z_0)$$

Let S_1 and S_2 be the level curves given by the equations $f = c_1$ and $g = c_2$, where $c_1 = f(x_0, y_0, z_0)$ and $c_2 = g(x_0, y_0, z_0)$. Let $C = S_1 \cap S_2$. The two surfaces that we have are

$$\frac{x+z}{y} = c_1 \qquad (c_1 \neq 0)$$
$$(x-y)^2 - z^2 = c_2$$

From the first equation, we get that $x + z = c_1 y$, which means $z = c_1 y - x$. Putting this value of z in the second equation, we get

$$(x-y)^2 - (c_1 y - x)^2 = c_2$$

and this implies

$$x = \frac{y^2(c_1^2 - 1) + c_2}{2y(c_1 - 1)} \quad , \quad z = \frac{y^2(c_1^2 - 2c_1 + 1) - c_2}{2y(c_1 - 1)} = \frac{y^2(c_1 - 1)^2 - c_2}{2y(c_1 - 1)}$$

Now, we also have the equation $\dot{y} = y$, which is again separable, and we also have the initial condition $y(0) = y_0$. Solving this DE, we get

$$\int \frac{1}{y}\, \mathrm{d}y = \int 1\, \mathrm{d}t = t + C$$

for some $C \in \mathbf{R}$. This gives us

$$\ln y = t + C$$

and hence we get that $C = \ln y_0$. This gives us

$$y = y_0 e^t$$

and hence

$$(x, y, z) = \left(\frac{y_0^2 e^{2t}(c_1^2 - 1) + c_2}{2y_0 e^t (c_1 - 1)}, y_0 e^t, \frac{y_0^2 e^{2t}(c_1 - 1)^2 - c_2}{2y_0 e^t (c_1 - 1)} \right)$$

where

$$c_1 = \frac{x_0 + z_0}{y_0} \quad , \quad c_2 = (x_0 - y_0)^2 - z_0^2$$

So, if $\boldsymbol{p_0} = (x_0, y_0, z_0)$ is our initial point, then the solution to the IVP is the function $\boldsymbol{\varphi_{p_0}}$ defined by

$$\boldsymbol{\varphi_{p_0}}(t) = \left(\frac{y_0^2 e^{2t}(c_1^2 - 1) + c_2}{2y_0 e^t (c_1 - 1)}, y_0 e^t, \frac{y_0^2 e^{2t}(c_1 - 1)^2 - c_2}{2y_0 e^t (c_1 - 1)} \right)$$

(c) From the parametric form of the solution, it is clear that the solution $\boldsymbol{\varphi_{p_0}}$ depends smoothly on $\boldsymbol{p_0} = (x_0, y_0, z_0)$: note that the constants c_1 and c_2 are $\mathscr{C}^1$ functions of (x_0, y_0, z_0), and hence the solution also depends on the point in a smooth way.

4). For $\boldsymbol{p} = (x, y, z) \in U$ and $t \in J_{\max}(\boldsymbol{p})$ we define

$$\boldsymbol{y_1}(t) = \left(e^t, 0, \left(\frac{1}{2} y e^{2t} - \frac{1}{2} y \right) \sec^2(\theta(t, x, y, z)) \right)$$
$$\boldsymbol{y_2}(t) = \left(0, e^t, \left(\frac{1}{2} x e^{2t} - \frac{1}{2} x \right) \sec^2(\theta(t, x, y, z)) \right)$$

(a) We show that $\boldsymbol{y_1}$ is a solution of the homogenous linear IVP

$$\dot{\boldsymbol{\zeta}} = A(t, x, y, z)\boldsymbol{\zeta} \quad , \quad \boldsymbol{\zeta}(0) = \boldsymbol{e_1}$$

where $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ is the standard basis of $\mathbf{R}^3$. We observe a couple of things. First note that

$$\frac{\partial \theta}{\partial t}(t, x, y, z) = xye^{2t}$$

Now, observe that

$$\boldsymbol{y_1'}(t) = \left(e^t, 0, ye^{2t}\sec^2(\theta(t, x, y, z)) + \left(\frac{1}{2} y e^{2t} - \frac{1}{2} y \right) \frac{\partial \theta}{\partial t} 2\sec^2(\theta(t, x, y, z))\tan(\theta(t, x, y, z)) \right)$$
$$= \left(e^t, 0, ye^{2t}\sec^2(\theta(t, x, y, z)) + 2xye^{2t} \left(\frac{1}{2} y e^{2t} - \frac{1}{2} y \right) \sec^2(\theta(t, x, y, z))\tan(\theta(t, x, y, z)) \right)$$

Now, observe that

$$A(t,x, y, z)\boldsymbol{y_1}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ye^t\sec^2(\theta(t, x, y, z)) & xe^t\sec^2(\theta(t, x, y, z)) & 2xye^{2t}\tan(\theta(t, x, y, z)) \end{bmatrix} \boldsymbol{y_1}(t)$$
$$= \left(e^t, 0, ye^{2t}\sec^2(\theta(t, x, y, z)) + 2xye^{2t} \left(\frac{1}{2} y e^{2t} - \frac{1}{2} y \right) \sec^2(\theta(t, x, y, z))\tan(\theta(t, x, y, z)) \right)$$

and hence we obtain

$$\boldsymbol{y_1'}(t) = A(t, x, y, z)\boldsymbol{y_1}(t)$$

Now, observe that

$$\boldsymbol{y_1}(0) = (1, 0, 0) = \boldsymbol{e_1}$$

and so this proves that $\boldsymbol{y_1}$ is a solution of the given IVP.

(b) First, observe that

$$\frac{\partial \boldsymbol{\varphi}}{\partial y}(x, y, z, t) = \left(0, e^t, \left(\frac{1}{2} x e^{2t} - \frac{1}{2} x \right) \sec^2(\theta(t, x, y, z)) \right) = \boldsymbol{y_2}(t)$$

So, we see that

$$\dot{\boldsymbol{y_2}}(t) = \frac{\partial \boldsymbol{\varphi}}{\partial t \partial y}(x, y, z, t) = \frac{\partial \boldsymbol{\varphi}}{\partial y \partial t}(x, y, z, t)$$

where above we used equality of mixed partial derivatives, since $\boldsymbol{\varphi}$ is a $\mathscr{C}^2$ function. Now, observe that

$$\frac{\partial \boldsymbol{\varphi}}{\partial y \partial t}(x,y,z,t) = \frac{\partial \dot{\boldsymbol{\varphi}}}{\partial y}(x,y,z,t) = \frac{\partial}{\partial y}\boldsymbol{v}(\boldsymbol{\varphi}(x,y,z,t))$$

To compute the above partial derivative, we will use the chain rule. Consider the function $\boldsymbol{v} \circ \boldsymbol{\varphi}$. Note that

$$\frac{\partial}{\partial y}\boldsymbol{v} \circ \boldsymbol{\varphi}(x,y,z,t) = \text{second column of } J(\boldsymbol{v} \circ \boldsymbol{\varphi})(x,y,z,t)$$

Also by the chain rule,

$$\begin{aligned} J(\boldsymbol{v} \circ \boldsymbol{\varphi})(x,y,z,t) &= (J\boldsymbol{v})(\boldsymbol{\varphi}(x,y,z,t)) \cdot (J\boldsymbol{\varphi})(x,y,z,t) \\ &= A(t,x,y,z) \cdot (J\boldsymbol{\varphi})(x,y,z,t) \end{aligned}$$

Now, the second column of the matrix $(J\boldsymbol{\varphi})(x,y,z,t)$ is simply $\frac{\partial \boldsymbol{\varphi}}{\partial y}(x,y,z,t) = \boldsymbol{y_2}(t)$, as we saw above. So it follows that the second column of $J(\boldsymbol{v} \circ \boldsymbol{\varphi})(x,y,z,t)$ is $A(t,x,y,z)\boldsymbol{y_2}(t)$. So, combining everything, we see that

$$\dot{\boldsymbol{y}}_{\boldsymbol{2}}(t) = A(t,x,y,z)\boldsymbol{y_2}(t)$$

Also, observe that

$$\boldsymbol{y_2}(0) = (0,1,0) = \boldsymbol{e_2}$$

and hence this shows that $\boldsymbol{y_2}$ is the solution of the given IVP, and this completes the proof.

(c) As in part **(b)** above, the solution to the IVP

$$\dot{\boldsymbol{\zeta}} = A(t,x,y,z)\boldsymbol{\zeta} \quad , \quad \boldsymbol{\zeta}(0) = \boldsymbol{e_3}$$

will be

$$\boldsymbol{y_3}(t) = \frac{\partial \boldsymbol{\varphi}}{\partial y}(x,y,z,t)$$

and the justification will be similar to what we did in part **(b)**.