HW-8

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1). First, we show that Φ and Ψ are well defined, i.e if $e(\theta_1) = e(\theta_2)$ then

$$\Phi(e(\theta_1)) = \Phi(e(\theta_2)) \quad , \quad \Psi(e(\theta_1)) = \Psi(e(\theta_2))$$

Let us first show the equality for Φ. So, let $\theta_1, \theta_2 \in \mathbf{R}$ be such that $e(\theta_1) = e(\theta_2) \neq -1$. Since e has period 2π, we see that

$$\theta_1 = \theta_2 \pmod{2\pi}$$

Suppose $\theta \in (-\pi, \pi)$ is such that

$$\theta = \theta_1 = \theta_2 \pmod{2\pi}$$

and clearly we see that $e(\theta) = e(\theta_1) = e(\theta_2)$ (note that such a θ exists because $e(\theta_1) = e(\theta_2) \neq -1$). Because $\theta = \theta_1 \pmod{2\pi}$, we can write $\theta_1 = \theta + 2\pi k$ for some $k \in \mathbb{Z}$. Then

$$\Phi(e(\theta_1)) = \tan(\theta_1/2) = \tan(\theta/2 + k\pi) = \tan(\theta/2) = \Phi(e(\theta))$$

since tan has period π. By the same reasoning, we have that $\Phi(e(\theta_2)) = \Phi(e(\theta))$, and hence

$$\Phi(e(\theta_1)) = \Phi(e(\theta_2))$$

showing that Φ is well-defined.

Let us do the same thing for Ψ. So, let $\theta_1, \theta_2 \in \mathbf{R}$ be such that $e(\theta_1) = e(\theta_2) \neq 1$. Since e has period 2π we see that

$$\theta_1 = \theta_2 \pmod{2\pi}$$

Suppose $\theta \in (0, 2\pi)$ is such that

$$\theta = \theta_1 = \theta_2 \pmod{2\pi}$$

and clearly we see that $e(\theta) = e(\theta_1) = e(\theta_2)$ (and again, note that such a θ exists because $e(\theta_1) = e(\theta_2) \neq 1$). Because $\theta = \theta_1 \pmod{2\pi}$, we can write $\theta_1 = \theta + 2\pi k$ for some $k \in \mathbb{Z}$. Then

$$\Psi(e(\theta_1)) = \cot(\theta_1/2) = \cot(\theta/2 + k\pi) = \cot(\theta/2) = \Psi(e(\theta))$$

since cot has period π. By the same reasoning, we have that $\Psi(e(\theta_2)) = \Psi(e(\theta))$, and hence

$$\Psi(e(\theta_1)) = \Psi(e(\theta_2))$$

showing that Ψ is well-defined.

Next, we will show that the map

$$\psi \circ \Phi^{-1} : \mathbf{R} \setminus \{0\} \to \mathbf{R} \setminus \{0\}$$

is the map $y \mapsto 1/y$. So, let $y \in \mathbf{R} \setminus \{0\}$. Observe that $e((-\pi, \pi)) = \boldsymbol{S}^1 \setminus \{-1\}$. So, let $\theta \in (-\pi, \pi)$ be such that $e(\theta) = \Phi^{-1}(y)$, which implies that

$$\tan(\theta/2) = y$$

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and since $y \neq 0$, we see that $\theta \neq 0$, and hence $e(\theta) \neq 1 \in \boldsymbol{S}^1$, which means that $\Psi(e(\theta))$ makes sense. So, we have

$$(\Psi \circ \Phi^{-1})(y) = \Psi(e(\theta)) = \cot(\theta/2) = \frac{1}{\tan(\theta/2)} = \frac{1}{y}$$

and this proves the claim.

2). Consider the vector field $\boldsymbol{v}(e(\theta)) = (\cos\theta + \sin\theta - 1)\frac{\mathrm{d}}{\mathrm{d}\theta}$ on $\boldsymbol{S}^1$. Let $f : \mathbf{R} \to \mathbf{R}$ be the map $y \mapsto y(1-y)$ and $g : \mathbf{R} \to \mathbf{R}$ be the map $z \mapsto 1 - z$. We show that

$$\Phi_*(\boldsymbol{v}|_{U_{-1}}) = f\vartheta \tag{0.1}$$

$$\Psi_*(\boldsymbol{v}|_{U_1}) = g\vartheta \tag{0.2}$$

Let us first show (0.1). So, suppose $y_0 \in \mathbf{R}$, and let $\theta_0 \in (-\pi, \pi)$ be such that $e(\theta_0) = \Phi^{-1}(y_0)$. Equality (0.1) is an equality of vector fields, so we need to show that

$$\Phi_*(\boldsymbol{v}|_{U_{-1}}(e(\theta_0))) = f(y_0)\vartheta(y_0) = y_0(1-y_0)\vartheta(y_0)$$

Note that the above equality is an equality of derivations, because Φ_* is a map from derivations to derivations. Suppose $g_{y_0} \in \mathscr{C}^\infty_{y_0}$, i.e g_{y_0} is a germ of $\mathscr{C}^\infty$ functions at y_0 represented by (g, U), where U is some open set in $\mathbf{R}$. Define $f = g \circ \Phi$ on $U' = \Phi^{-1}(U)$, and we get a germ $f_{e(\theta_0)} \in \mathscr{C}^\infty_{e(\theta_0)}$ represented by (f, U'). By definition, we have

$$\Phi_*(\boldsymbol{v}|_{U_{-1}}(e(\theta_0)))(g_{y_0}) = \boldsymbol{v}|_{U_{-1}}(e(\theta_0))(f_{e(\theta_0)})$$

and hence showing equality (0.1) is equivalent to showing

$$\boldsymbol{v}|_{U_{-1}}(e(\theta_0))(f_{e(\theta_0)}) = y_0(1-y_0)\vartheta(y_0)(g_{y_0}) \tag{0.3}$$

Now observe that

$$\begin{aligned}
\boldsymbol{v}|_{U_{-1}}(e(\theta_0))(f) &= (\cos\theta_0 + \sin\theta_0 - 1)\left.\frac{\mathrm{d}(f\circ e)}{\mathrm{d}\theta}\right|_{\theta=\theta_0} \\
&= (\cos\theta_0 + \sin\theta_0 - 1)\left.\frac{\mathrm{d}(g\circ\Phi\circ e)}{\mathrm{d}\theta}\right|_{\theta=\theta_0} \\
&= (\cos\theta_0 + \sin\theta_0 - 1)\left.\frac{\mathrm{d}(g(\tan(\theta/2)))}{\mathrm{d}\theta}\right|_{\theta=\theta_0} \\
&= (\cos\theta_0 + \sin\theta_0 - 1)\frac{\mathrm{d}g}{\mathrm{d}y}(\tan(\theta_0/2))\frac{\mathrm{d}\tan(\theta/2)}{\mathrm{d}\theta}|_{\theta=\theta_0} \\
&= (\cos\theta_0 + \sin\theta_0 - 1)\frac{\mathrm{d}g}{\mathrm{d}y}(y_0)\cdot\frac{1}{2}\sec^2(\theta_0/2) \\
&= \frac{1}{2}(\cos\theta_0 + \sin\theta_0 - 1)\sec^2(\theta_0/2)\vartheta(y_0)(g)
\end{aligned}$$

Now, we use the identities

$$\begin{aligned}
\sin(\theta_0) &= \frac{2\tan(\theta_0/2)}{1+\tan^2(\theta_0/2)} = \frac{2y_0}{1+y_0^2} \\
\cos(\theta_0) &= \frac{1-\tan^2(\theta_0/2)}{1+\tan^2(\theta_0/2)} = \frac{1-y_0^2}{1+y_0^2}
\end{aligned}$$

and hence we get

$$\boldsymbol{v}|_{U_{-1}}(e(\theta_0))(f) = \frac{1}{2}\left(\frac{1-y_0^2+2y_0}{1+y_0^2} - 1\right)(1+y_0^2)\vartheta(y_0)(g) = y_0(1-y_0)\vartheta(y_0)(g)$$

and the above equality shows equation (0.3), and hence proves equation (0.1).

Similarly, we now prove equation (0.2). Again, let $y_0 \in \mathbf{R}$, and let $\theta_0 \in (0, 2\pi)$ be such that $e(\theta_0) = \Psi^{-1}(y_0)$. Again, equality (0.2) is an equality of vector fields, and so we need to show that

$$\Psi_*(\boldsymbol{v}|_{U_1}(e(\theta_0))) = g(y_0)\vartheta(y_0) = (1-y_0)\vartheta(y_0)$$

Again, note that the above equality is an equality of derivations, since Ψ_* is a map from derivations to derivations. To show the above equality, let $g_{y_0} \in \mathscr{C}^\infty_{y_0}$, i.e g_{y_0} is a germ of $\mathscr{C}^\infty$ functions at y_0 represented by (g, U), where U is an open set in $\mathbf{R}$. Let $f = g \circ \Psi$ on $U' = \Psi^{-1}(U)$, and we get a germ $f_{e(\theta_0)} \in \mathscr{C}^\infty_{e(\theta_0)}$ represented by (f, U'). By definition, we have

$$\Psi_*(\boldsymbol{v}|_{U_1}(e(\theta_0)))(g_{y_0}) = \boldsymbol{v}|_{U_1}(e(\theta_0))(f_{e(\theta_0)})$$

and hence showing equality (0.2) is equivalent to showing

$$\boldsymbol{v}|_{U_1}(e(\theta_0))(f_{e(\theta_0)}) = (1-y_0)\vartheta(y_0)(g_{y_0}) \tag{0.4}$$

Now observe that

$$\begin{aligned}
\boldsymbol{v}|_{U_1}(e(\theta_0))(f) &= (\cos\theta_0 + \sin\theta_0 - 1)\left.\frac{\mathrm{d}(f\circ e)}{\mathrm{d}\theta}\right|_{\theta=\theta_0} \\
&= (\cos\theta_0 + \sin\theta_0 - 1)\left.\frac{\mathrm{d}(g\circ\Psi\circ e)}{\mathrm{d}\theta}\right|_{\theta=\theta_0} \\
&= (\cos\theta_0 + \sin\theta_0 - 1)\left.\frac{\mathrm{d}g(\cot(\theta/2))}{\mathrm{d}\theta}\right|_{\theta=\theta_0} \\
&= (\cos\theta_0 + \sin\theta_0 - 1)\frac{\mathrm{d}g}{\mathrm{d}y}(\cot(\theta_0/2))\frac{\mathrm{d}\cot(\theta/2)}{\mathrm{d}\theta}|_{\theta=\theta_0} \\
&= (\cos\theta_0 + \sin\theta_0 - 1)\frac{\mathrm{d}g}{\mathrm{d}y}(y_0)\frac{1}{2}(-\mathrm{cosec}^2(\theta_0/2)) \\
&= -\frac{1}{2}(\cos\theta_0 + \sin\theta_0 - 1)(1+\cot^2(\theta_0/2))\vartheta(y_0)(g) \\
&= -\frac{1}{2}(\cos\theta_0 + \sin\theta_0 - 1)(1+y_0^2)\vartheta(y_0)(g)
\end{aligned}$$

Now, we use the identities

$$\begin{aligned}
\sin(\theta_0) &= \frac{2\cot(\theta_0/2)}{1+\cot^2(\theta_0/2)} = \frac{2y_0}{1+y_0^2} \\
\cos(\theta_0) &= \frac{\cot^2(\theta_0/2)-1}{\cot^2(\theta_0/2)+1} = \frac{y_0^2-1}{y_0^2+1}
\end{aligned}$$

and hence we get

$$\boldsymbol{v}|_{U_1}(e(\theta_0))(f) = -\frac{1}{2}\left(\frac{y_0^2-1+2y_0}{y_0^2+1} - 1\right)(y_0^2+1)\vartheta(y_0)(g) = (1-y_0)\vartheta(y_0)(g)$$

and the above equality shows equation (0.4), and hence it shows equation (0.2). This completes the proof.

3). To show that $g^t(e(\theta))$ is well-defined, it is enough to show that if $e(\theta_1) = e(\theta_2)$, then

$$\frac{e^t\tan(\theta_1/2)}{e^t\tan(\theta_1/2) - \tan(\theta_1/2) + 1} = \frac{e^t\tan(\theta_2/2)}{e^t\tan(\theta_2/2) - \tan(\theta_2/2) + 1}$$

and note that to show the above equation, it is enough to show that

$$\tan(\theta_1/2) = \tan(\theta_2/2)$$

But this is easy: because e has period 2π and $e(\theta_1) = e(\theta_2)$, we see that

$$\theta_2 = \theta_1 + 2\pi k$$

for some $k \in \mathbb{Z}$. This will imply that

$$\tan(\theta_2/2) = \tan(\theta_1/2 + \pi k) = \tan(\theta_1/2)$$

because tan has period π. So, this shows that $g^t(e(\theta))$ is well-defined.

4). We now show that $\{g^t\}$ is a 1-parameter group of diffeomorphisms on $\boldsymbol{S}^1$. First, consider the DE

$$\dot{p} = \boldsymbol{v}(p)$$

on $\boldsymbol{S}^1$. We first restrict this DE to U_{-1}, and we transform this DE to a DE in $\mathbf{R}$ via the map Φ. So, let $\boldsymbol{w}$ be the map on $\mathbf{R}$ defined as follows

$$\boldsymbol{w}(y) = \Phi'(e(\theta))\boldsymbol{v}(e(\theta))$$

where $e(\theta) = \Phi^{-1}(y) \in U_{-1}$. Then as proven in Lecture 18, the DE $\dot{p} = \boldsymbol{v}(p)$ is equivalent to the DE $\dot{q} = \boldsymbol{w}(q)$, in the sense that given a solution to $\dot{q} = \boldsymbol{w}(q)$, a solution of $\dot{p} = \boldsymbol{v}(p)$ can be obtained by pulling back via Φ. Now, observe that

$$\begin{aligned}
\boldsymbol{w}(y) &= \Phi'(e(\theta))\boldsymbol{v}(e(\theta)) \\
&= \frac{\mathrm{d}}{\mathrm{d}\theta}\tan(\theta/2)(\cos\theta + \sin\theta - 1) \\
&= \frac{1}{2}\sec^2(\theta/2)(\cos\theta + \sin\theta - 1) \\
&= \frac{1}{2}(1 + y^2)\left(\frac{1 - y^2 + 2y}{1 + y^2} - 1\right) \\
&= y(1 - y)
\end{aligned}$$

So, consider the DE

$$\dot{q} = \boldsymbol{w}(q) = q(1 - q)$$

on $\mathbf{R}$. This is a separable DE, which we know how to solve.

$$\int \frac{1}{q(1-q)}\,\mathrm{d}q = t + C'$$

for some $C' \in \mathbf{R}$ and solving this DE, we get

$$\frac{q}{1-q} = e^{t+C'}$$

and solving for q, we get

$$q = \frac{e^{t+C'}}{e^{t+C'} + 1} = \frac{Ce^t}{Ce^t + 1}$$

where $C = e^{C'}$. If we put $q(0) = q_0$, then we see that

$$C = \frac{q_0}{1 - q_0}$$

and hence

$$q = \frac{q_0 e^t}{q_0 e^t + 1 - q_0}$$

So we consider the following:

$$h^t x = \frac{xe^t}{xe^t + 1 - x} \quad , \quad x, t \in \mathbf{R}$$

In problem **5)** of HW-5, we have already shown that this is a one-parameter group of diffeomorphisms. If $x \in \{0, 1\}$, then the phase-flow will be a constant map. If x is in one of the connected components $(-\infty, 0)$, $(0, 1)$ or $(1, \infty)$, then we get a phase-flow which stays in the component of x.

Now, consider the pullback under Φ of the one parameter group defined above, i.e consider

$$g^t e(\theta) = e\left\{2 \arctan\left(\frac{\tan(\theta/2)e^t}{\tan(\theta/2)e^t + 1 - \tan(\theta/2)}\right)\right\}$$

Since h^t is a one-parameter group, it follows that g^t is also a one-parameter group of diffeomorphisms. This is because, as maps, it is true that

$$g^t = \Phi^{-1} \circ h^t \circ \Phi$$

for all $t \in \mathbf{R}$, and since h^t satisfies the axioms for a one-parameter group, it follows that g^t also satisfies the axioms of a one-parameter group.

Since we have pulled back via the map Φ, the one parameter group g^t is only defined on U_{-1}. Now, observe that the formula for g^t actually makes sense over all of $\boldsymbol{S}^1$, i.e $g^t(e(\theta))$ is well-defined for all θ. Hence, it follows that g^t is actually a one-parameter group of diffeomorphisms on $\boldsymbol{S}^1$.

5). In this problem, we determine the orbits of $\{g^t\}$ and the fixed points of $\{g^t\}$.

Fixed Points. Suppose $e(\theta) \in S^1$ is a fixed point of $\{g^t\}$. This means that

$$g^t e(\theta) = e(\theta)$$

for all $t \in \mathbf{R}$. So, this means that the phase flow over S^1 defined for this initial point is a constant map. This means that the push-forward under Φ of this phase flow in $\mathbf{R}$ is a constant solution to the DE

$$\dot{q} = q(1 - q)$$

Hence, it follows that $\Phi(e(\theta)) \in \{0, 1\}$, because only the initial points $0, 1$ have a constant phase flow. This means that

$$\tan(\theta/2) \in \{0, 1\}$$

and hence if we restrict $\theta \in (-\pi, \pi)$, we see that $\theta \in \{0, \pi/2\}$. So, it follows that the fixed points are

$$(\cos(0), \sin(0)) = (1, 0) \text{ and } (\cos(\pi/2), \sin(\pi/2)) = (0, 1)$$

Orbits. Since the points $(0, 1)$ and $(1, 0)$ are fixed points of $\{g^t\}$, they form separate orbits. We will now determine the remaining orbits.

Suppose $e(\theta_1)$ and $e(\theta_2)$ are two points in $\boldsymbol{S}^1 \setminus \{-1\}$ in the same orbit of $\{g^t\}$, i.e $e(\theta_2) = g^t e(\theta_1)$ for some $t \in \mathbf{R}$. Taking their images under Φ, we see that

$$\Phi(e(\theta_2)) = h^t \circ \Phi(e(\theta_1))$$

Now, as we remarked in problem **4)**, both the points $\Phi(e(\theta_1))$ and $\Phi(e(\theta_2))$ must belong to one of the connected components $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$ (since the phase flow always remains in the same component).

Conversely, if we take any two points $e(\theta_1)$ and $e(\theta_2)$ in $\boldsymbol{S}^1 \setminus \{-1\}$ such that $\Phi(e(\theta_1))$ and $\Phi(e(\theta_2))$ belong to the same connected component, without loss of generality say both the points are in $(0, 1)$, then we claim that $\Phi(e(\theta_1))$ and $\Phi(e(\theta_2))$ are in the same orbit of $\{h^t\}$. This easily follows by solving for t in the equation

$$h^t\Phi(e(\theta_1)) = \Phi(e(\theta_2))$$

So, pulling back under Φ, we see that $e(\theta_1)$ and $e(\theta_2)$ belong to the same orbit of $\{g^t\}$. Now, the space $(\boldsymbol{S}^1 \setminus \{-1\}) \setminus \{(1, 0), (0, 1)\}$ has three connected components, all of which are open arcs. So, it follows that any two points on $\boldsymbol{S}^1 \setminus \{-1\}$ which belong to any one of these open arcs lie in the same orbit of $\{g^t\}$. So, there are atmost 5 orbits of $\{g^t\}$ on $\boldsymbol{S}^1 \setminus \{-1\}$.

Now we deal with the point $-1 \in \boldsymbol{S}^1$. Observe that by assumption, this point is given by $-1 = e(-\pi) = e(\pi)$, and this point corresponds to either $-\infty$ or ∞ on the extended real line. Note that if $x \in (1, \infty)$ or $x \in (-\infty, 0)$, then there is some $t \in \mathbf{R}$ for which

$$xe^t + 1 - x = 0$$

This means that the denominator of h^tx will be 0 for some $t \in \mathbf{R}$, meaning that $h^tx = \infty$ or $-\infty$ for some $t \in \mathbf{R}$, depending upon whether $x \in (1, \infty)$ or $x \in (-\infty, 0)$. Pulling back by Φ, this means that the points $\Phi^{-1}(x)$ and $-1 \in \boldsymbol{S}^1$ belong to the same orbit of $\{h\}^t$. So, it follows that the one-parameter group $\{h\}^t$ on $\boldsymbol{S}^1$ has only four orbits: two singleton points $\{(1, 0)\}$ and $\{(0, 1)\}$, and the two remaining open arcs on the circle.