HW-9

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1). Here, we solve problem **23** of Cookbook-I.

23 $y' = (1-2x)y^2, \quad y(0) = \frac{-1}{6}$

Solution. This is a separable DE. We have

$$\frac{1}{y^2}y' = (1-2x)$$

Integrating both sides, we get

$$\int \frac{1}{y^2}\,\mathrm{d}y = \int 1 - 2x\,\mathrm{d}x$$

and this gives us

$$\frac{-1}{y} = x - x^2 + C$$

where $C \in \mathbb{R}$ is some constant. Using the given initial condition, we get $C = 6$. Hence, the solution of the DE is

$$y = \frac{1}{x^2 - x - 6}$$

Note that y is defined on $\mathbf{R}$ minus the roots of the given polynomial. Observe that

$$x^2 - x - 6 = (x+2)(x-3)$$

and hence y is defined on $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$. Since the initial point $-1/6$ lies in the interval $(-2, 3)$, it follows that the interval of existence is $(-2, 3)$. ■

2). Let $(\tau, a) \in \mathbf{R}^2$. We will find a formula for $\varphi_{(\tau,a)}(t)$ for $t \in J(\tau, a)$. First, suppose $a \neq 0$. As in problem **1)**, we then see that

$$\frac{-1}{a} = \frac{-1}{\varphi_{(\tau,a)}(\tau)} = \tau - \tau^2 + C$$

which gives us

$$C = \tau^2 - \tau - \frac{1}{a}$$

and hence the formula $\varphi_{(\tau,a)}(t)$ is given by

$$\frac{-1}{\varphi_{(\tau,a)}(t)} = t - t^2 + \tau^2 - \tau - \frac{1}{a}$$

and solving this further, we obtain

$$\varphi_{(\tau,a)}(t) = \frac{1}{t^2 - t - \tau^2 + \tau + \frac{1}{a}} = \frac{a}{a(t^2 - t - \tau^2 + \tau) + 1}$$

Now, above if we put $a = 0$, we get the constant solution $\varphi_{(\tau,0)} \equiv 0$. So, the formula makes sense for all (τ, a), and hence this is the required formula.

Date: 16 April 2021.

3). Suppose (τ, a) is on the τ-axis. This means that $a = 0$. In this case, note that the zero function $\varphi_{(\tau,0)} \equiv 0$ is a solution to $(\Delta)_{(\tau,a)}$. Hence, the interval of existence in this case is $\mathbf{R}$, i.e $J(\tau, 0) = \mathbf{R}$.

4). Note that the curve $a(2\tau - 1)^2 = 4$ divides the plane into two domains (open connected sets), namely

$$\{(\tau, a) \mid a(2\tau - 1)^2 > 4\} \text{ and } \{(\tau, a) \mid a(2\tau - 1)^2 < 4\}$$

Since the origin $(0, 0)$ belongs to the second region above, it follows that region ① is contained in the second set above. So, a set theoretic description of ① is as follows:

$$① = \{(\tau, a) \mid a(2\tau - 1)^2 < 4, \quad a > 0\}$$

So, suppose the point (τ, a) is lies in ①. We know that

$$\varphi_{(\tau,a)}(t) = \frac{a}{a(t^2 - t - \tau^2 + \tau) + 1}$$

We claim that the denominator vanishes for no value of t, i.e the equation

$$a(t^2 - t - \tau^2 + \tau) + 1 = 0$$

has no solution $t \in \mathbf{R}$. This will imply that $J(\tau, a) = \mathbf{R}$, and that will complete the proof.

Since $a > 0$, the equation can be written as

$$t^2 - t - \tau^2 + \tau + \frac{1}{a} = 0$$

This is a quadratic equation with discriminant

$$D = 1 - 4\left(-\tau^2 + \tau + \frac{1}{a}\right)$$

Now, observe that

$$\begin{aligned}
&a(2\tau - 1)^2 < 4 \\
\implies &(2\tau - 1)^2 - \frac{4}{a} < 0 \\
\implies &4\tau^2 - 4\tau - \frac{4}{a} + 1 < 0 \\
\implies &\tau^2 - \tau - \frac{1}{a} + \frac{1}{4} < 0 \\
\implies &\frac{1}{4} < -\tau^2 + \tau + \frac{1}{a} \\
\implies &1 < 4\left(-\tau^2 + \tau + \frac{1}{a}\right) \\
\implies &D < 0
\end{aligned}$$

and hence the quadratic has no real solutions. This proves that $J(\tau, a) = \mathbf{R}$, thereby completing our proof.

5. We show that if (τ, a) is in region ⃝2, ⃝2' or ⃝3, then $\delta(\tau, a) > 0$. This is easy to see. As in problem **4)**, consider the following two domains in $\mathbb{R}^2$, which the curve $a(2\tau - 1)^2 = 4$ divides the plane into.

$$\{(\tau, a) \mid a(2\tau - 1)^2 > 4\} \text{ and } \{(\tau, a) \mid a(2\tau - 1)^2 < 4\}$$

(1) Suppose $(\tau, a) \in$ ⃝2 or ⃝2'. Because the origin $(0, 0)$ lies in the second set above, we immediately see that both ⃝2 and ⃝2' are subsets of $\{(\tau, a) \mid a(2\tau - 1)^2 > 4\}$. Also, note that $a > 0$, since $(\tau, a) \in$ ⃝2 or ⃝2'. So, we see that

$$a^2(2\tau - 1)^2 > 4a$$

and hence $\delta(\tau, a) > 0$ in this case.

(2) In this second case, suppose $(\tau, a) \in$ ⃝3. Again, since the origin $(0, 0)$ lies in the second set above, we see that ⃝3 is a subset of $\{(\tau, a) \mid a(2\tau - 1)^2 < 4\}$. Also, since $(\tau, a) \in$ ⃝3, we have that $a < 0$. So, multiplying the inequality $a(2\tau - 1)^2 < 4$ by a, which is a negative number, we get

$$a^2(2\tau - 1)^2 > 4a$$

and hence again $\delta(\tau, a) > 0$.

This completes the proof.

6). Suppose $(\tau, a) \in$ ⃝2. Again, consider the formula for $\varphi_{(\tau,a)}$ that we found before:

$$\varphi_{(\tau,a)}(t) = \frac{a}{a(t^2 - t - \tau^2 + \tau) + 1}$$

By the remarks made in problem **5)** above, we note that the set-theoretic description of the set ⃝2 is the following.

$$⃝2 = \left\{(\tau, a) \mid a(2\tau - 1)^2 > 4, \tau > \frac{1}{2}\right\}$$

Now, consider the following quadratic equation in t.

$$a(t^2 - t - \tau^2 + \tau) + 1 = 0$$

which is the same as the equation

$$at^2 - at + a(-\tau^2 + \tau) + 1 = 0$$

The discriminant of this quadratic is

$$D = a^2 - 4a(-a\tau^2 + a\tau + 1) = a^2 + 4a^2\tau^2 - 4a^2\tau - 4a = a^2(2\tau - 1)^2 - 4a = \delta(\tau, a)$$

and hence the solutions of this quadratic are

$$\frac{a \pm \sqrt{\delta(\tau, a)}}{2a} = \frac{1}{2} \pm \frac{\sqrt{\delta(\tau, a)}}{2a} = \frac{1}{2} \pm \frac{\sqrt{\delta(\tau, a)}}{2|a|}$$

where above we have used the fact that $a = |a|$, since $(\tau, a) \in$ ⃝2. So, the function $\varphi_{(\tau,a)}$ is defined on

$$\left(-\infty, \frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right) \cup \left(\frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right) \cup \left(\frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \infty\right)$$

The interval of existence in this case will be one of the above three intervals, in which the number τ belongs. Because $a > 0$, we see that

$$a^2(2\tau - 1)^2 > a^2(2\tau - 1)^2 - 4a$$

which means

$$a^2(2\tau - 1)^2 > \delta(\tau, a)$$

Taking square roots, we get

$$a(2\tau - 1) > \sqrt{\delta(\tau, a)}$$

and hence

$$\tau > \frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}$$

which implies that the interval of existence in this case is

$$J(\tau, a) = \left(\frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \infty\right)$$

which proves the claim.

7). This problem has a very similar solution to that of problem **6).** Note that the set-theoretic description of ⓶' is the following.

$$\textcircled{2'} = \left\{(\tau, a) \mid a(2\tau - 1)^2 > 4, \tau < \frac{1}{2}\right\}$$

Again, the solutions of the quadratic

$$a(t^2 - t - \tau^2 + \tau) + 1 = 0$$

are

$$\frac{1}{2} \pm \frac{\sqrt{\delta(\tau, a)}}{2|a|}$$

where again we have used the fact that $a = |a|$, since $(\tau, a) \in \textcircled{2'}$. So, the function $\varphi_{(\tau,a)}$ is defined on the union

$$\left(-\infty, \frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right) \cup \left(\frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right) \cup \left(\frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \infty\right)$$

So, the interval of existence will be the interval in which the number τ belongs. Because $a > 0$, we see that

$$a^2(2\tau - 1)^2 > a^2(2\tau - 1)^2 - 4a = \delta(\tau, a)$$

which can be written as

$$a^2(1 - 2\tau)^2 > \delta(\tau, a)$$

Since $\tau < \frac{1}{2}$, we see that $(1 - 2\tau) > 0$. Taking square roots, we see that

$$a(1 - 2\tau) > \sqrt{\delta(\tau, a)}$$

which implies that

$$\tau < \frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}$$

and hence, the interval of existence in this case is

$$J(\tau, a) = \left(-\infty, \frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right)$$

which proves the claim.

8). We follow the same strategy as in problems **6)** and **7)**. Note that the set-theoretic description of ③ is the following.

$$③ = \{(\tau, a) \mid a(2\tau - 1)^2 < 4, a < 0\}$$

The solutions of the quadratic

$$a(t^2 - t - \tau^2 + \tau) + 1 = 0$$

are

$$\frac{1}{2} \pm \frac{\sqrt{\delta(\tau, a)}}{2|a|}$$

where we have used the fact that $|a| = -a$, since $a < 0$. So, the function $\varphi_{(\tau,a)}$ is defined on the union

$$\left(-\infty, \frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right) \cup \left(\frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right) \cup \left(\frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \infty\right)$$

and hence the interval of existence will be the interval in which τ belongs. We consider three cases.

(1) In the first case, suppose $\tau = \frac{1}{2}$. In that case, observe that

$$\tau = 0 \in \left(\frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right)$$

and hence the interval of existence in this case will be

$$J(0, a) = \left(\frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right)$$

(2) In the second case, suppose $\tau > \frac{1}{2}$, which means that $2\tau - 1 > 0$. Because $a < 0$, we see that

$$a^2(2\tau - 1)^2 < a^2(2\tau - 1)^2 - 4a = \delta(\tau, a)$$

Taking square roots on both sides (use the fact that $2\tau - 1 = |2\tau - 1|$ and $|a| = -a$), we get

$$|a|(2\tau - 1) < \sqrt{\delta(\tau, a)}$$

from which we get

$$-a(2\tau - 1) < \sqrt{\delta(\tau, a)}$$

Dividing both sides by $-a$, which is positive, we get

$$2\tau - 1 < \frac{\sqrt{\delta(\tau, a)}}{-a} = \frac{\sqrt{\delta(\tau, a)}}{|a|}$$

and hence we get

$$\tau < \frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}$$

Also, because $a < 0$, $2\tau - 1 > 0$ and $\delta(\tau, a) > 0$, we get

$$a(2\tau - 1) < \sqrt{\delta(\tau, a)}$$

and dividing throughout by a, which is negative, we get

$$2\tau - 1 > \frac{\sqrt{\delta(\tau, a)}}{a}$$

and from here we obtain

$$\tau > \frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}$$

which implies that

$$\tau \in \left(\frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right)$$

and hence, in this case as well, we see that the interval of existence is

$$J(\tau, a) = \left(\frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right)$$

(3) In the final case, we assume that $\tau < \frac{1}{2}$, which implies that $2\tau - 1 < 0$. This case is handled similar to case (2), and even here, we obtain that the interval of existence is

$$J(\tau, a) = \left(\frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right)$$

So, the interval of existence has been found to be

$$J(\tau, a) = \left(\frac{1}{2} - \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}\right)$$

for all points (τ, a) in ③.

9). The strategy here will be the same as in problems **6)**, **7)** and **8)**. Note that if (τ, a) is on the right branch of the curve $a(2\tau - 1)^2 = 4$, then we have $a > 0$ and $\tau > \frac{1}{2}$. Also, we see that

$$\delta(\tau, a) = a^2(2\tau - 1)^2 - 4a = a[a(2\tau - 1)^2 = 4] = 0$$

In this case, the only solution of the quadratic

$$a(t^2 - t - \tau^2 + \tau) + 1 = 0$$

is the point

$$t = \frac{1}{2}$$

and hence the function $\varphi_{(\tau,a)}$ is defined on the union

$$\left(-\infty, \frac{1}{2}\right) \left(\frac{1}{2}, \infty\right)$$

By assumption, we have $\tau > \dfrac{1}{2}$, and hence the interval of existence in this case is

$$J(\tau, a) = \left(\frac{1}{2}, \infty\right)$$

10). The solution is similar to problem **9)**. If (τ, a) is on the left branch of the curve $a(2\tau 01)^2 = 4$, then we have $a > 0$ and $\tau < \dfrac{1}{2}$. Again, we have

$$\delta(\tau, a) = a^2(2\tau - 1)^2 - 4a = a[a(2\tau - 1)^2 = 4] = 0$$

and again the only solution of the quadratic

$$a(t^2 - t - \tau^2 + \tau) + 1 = 0$$

is the point

$$t = \frac{1}{2}$$

and hence the function $\varphi_{(\tau,a)}$ is defined on the union

$$\left(-\infty, \frac{1}{2}\right)\left(\frac{1}{2}, \infty\right)$$

By assumption, we have $\tau < \dfrac{1}{2}$, and hence the interval of existence in this case is

$$J(\tau, a) = \left(-\infty, \frac{1}{2}\right)$$

which proves the claim.

11). To show that $\tilde{\Omega}$ is the connected component of W containing the origin $\mathbf{0}$, we will show the following.

(1) $\mathbf{0} \in \tilde{\Omega}$.
(2) $\tilde{\Omega}$ is an open subset of W.
(3) $\tilde{\Omega}$ is a closed subset of W.
(4) $\tilde{\Omega}$ is path connected.

Note that (4) will imply that $\tilde{\Omega}$ is a connected set. Also, points (2) and (3) will imply that $\tilde{\Omega}$ cannot sit inside a bigger connected set, and hence $\tilde{\Omega}$ must be the connected component of $\mathbf{0}$.

First, note that $\mathbf{0} = (0, 0, 0) \in \tilde{\Omega}$: this is because $\varphi_{(0,0)} \equiv 0$, i.e $\varphi_{(0,0)}$ is the constant zero map, whose interval of existence is $\mathbf{R}$. In particular, we have that $0 \in J(0, 0)$, and hence by definition, $(0, 0, 0) \in \tilde{\Omega}$. This proves (1).

To prove (2), note that $\tilde{\Omega}$ is an open subset of $\mathbf{R}^3$ by Proposition 2.1.4 of Lecture 23. Also, W is an open subset of $\mathbf{R}^3$, because it is the complement of a closed set. So, it remains to show that $\tilde{\Omega} \subseteq W$, and that will prove (2). So, suppose $(t, \tau, a) \in \tilde{\Omega}$. Then, we have some cases to handle.

(1) In the first case, (τ, a) is in the τ-axis. We know by problem **3)** that in this case, $J(\tau, a) = \mathbf{R}$, which is equivalent to saying that the quadratic

$$a(t^2 - t - \tau^2 + \tau) + 1 = 0$$

has no solution in t. In that case, clearly we see that $(t, \tau, a) \notin S \cup F_1 \cup F_2$, and hence $(t, \tau, a) \in W$.

(2) In the second case, $(\tau, a) \in$ ①. By problem **4)**, we know that $J(\tau, a) = \mathbf{R}$, and hence again, the quadratic

$$a(t^2 - t - \tau^2 + \tau) + 1 = 0$$

has no solution in t. So, again we see that $(t, \tau, a) \notin S \cup F_1 \cup F_2$, which again implies that $(t, \tau, a) \in W$.

(3) In the third case, we have $(\tau, a) \in$ the closure of ②. By problems **6)** and **9)**, we know that

$$J(\tau, a) = \left(\frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}, \infty \right)$$

($\delta(\tau, a) = 0$ if (τ, a) lies on the right branch of the curve $a(2\tau - 1)^2 = 4$). By definition, we know that $t \in J(\tau, a)$, and hence

$$t > \frac{1}{2} + \frac{\sqrt{\delta(\tau, a)}}{2|a|}$$

Also, because t is in the interval of existence, this means that

$$a(t^2 - t - \tau^2 + \tau) + 1 \neq 0$$

since the above term is the denominator of $\varphi_{(\tau,a)}(t)$. All of this implies that $(t, \tau, a) \notin S \cup F_1 \cup F_2$, and hence $(t, \tau, a) \in W$.

(4) In the fourth case, we have $(\tau, a) \in$ the closure of ②'. Using a very similar argument as in point number (3) above and using problems **7)** and **10)**, it can be argued that $(t, \tau, a) \in W$.

(5) In the last case, we have that $(\tau, a) \in$ ③. Clearly, because t lies in the interval of existence, it followst that

$$a(t^2 - t - \tau^2 + \tau) + 1 \neq 0$$

as the above term is the denominator of $\varphi_{(\tau,a)}(t)$. So again, we see that $(t, \tau, a) \notin S \cup F_1 \cup F_2$, and hence $(t, \tau, a) \in W$.

So this completes the proof of (2), i.e $\tilde{\Omega}$ is an open subset of W (note that $\tilde{\Omega}$ is already known to be open in $\mathbf{R}^3$).

Next, we prove (3), i.e $\tilde{\Omega}$ is a closed subset of W. We show this by showing that $\tilde{\Omega}$ has no limit points in $W - \tilde{\Omega}$. Couldn't complete this part.

Finally, we show that $\tilde{\Omega}$ is path connected, and it is actually true that straight line paths do the job. Suppose (t_1, τ_1, a_1) and (t_2, τ_2, a_2) are two points in $\tilde{\Omega}$. Couldn't complete this part.