DISCRETE MATH PROBLEMS

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Abstract. These are solutions to problems in discrete mathematics. I solved these while taking a course in discrete math.

1. Assignment-2

(1). We will show that $\alpha \in \mathbb{R}$ is irrational if and only if there are sequences $\{a_n\}$ and $\{b_n\}$ of integers such that

$$\lim_{n\to\infty} b_n\alpha - a_n = 0$$

First, suppose that such sequences $\{a_n\}$ and $\{b_n\}$ exist. For the sake of contradiction, assume that α is rational, and let $\alpha = \frac{p}{q}$, where $\gcd(p, q) = 1$ and $q > 0$. Then, we have

$$\lim_{n\to\infty} \left| \frac{b_n p}{q} - a_n \right| = 0$$

Let $\epsilon > 0$ be arbitrary. Then, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left| \frac{b_n p}{q} - a_n \right| < \frac{\epsilon}{q}$$

which implies that

$$|b_n p - a_n q| < \epsilon$$

for all $n \geq N$. Now, observe that $b_n p - a_n q \in \mathbb{Z}$, and hence $|b_n p - a_n q| \geq 1$. But, since ϵ was arbitrary, this is a contradiction. Hence, α must be irrational.

Conversely, suppose α is irrational. Let $n \in \mathbb{N}$ be fixed. Then, we know that one of the numbers in $\{\alpha, 2\alpha, ..., n\alpha\}$ is within $\frac{1}{n}$ of some integer. So, let $b_n\alpha$ (where $1 \leq b_n \leq n$) be within $\frac{1}{n}$ of the integer a_n. This can be written as

$$0 \leq |b_n\alpha - a_n| < \frac{1}{n}$$

where the quantity is strictly greater than 0 because α is assumed to be irrational. Now, since $\frac{1}{n} \to 0$ as $n \to \infty$, we see that

$$|b_n\alpha - a_n| \to 0 \text{ as } n \to \infty$$

and hence the required sequences b_n and a_n have been found. This completes the proof.

Using this, we will now show that $\sqrt{2}$ is irrational.

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(2). First, we will prove the statement when all $x_1, ..., x_n$ are non-negative. So, suppose $x_1, ..., x_n$ are real numbers such that $0 \le x_i \le 1$ for all $1 \le i \le n$. If $x_i = x_j$ for some $i \ne j$, then the statement is trivial, because

$$|x_i - x_j| = 0 < \frac{n}{2^n - 1}$$

and if we set a_k to be 1 if $k = i$, -1 if $k = j$ and 0 otherwise, we see that

$$\left|\sum_k a_k x_k\right| = 0 < \frac{n}{2^n - 1}$$

So, let us assume that each x_i is distinct.

Now, consider the set $S = \{x_1, ..., x_n\}$. There are $2^n - 1$ non-empty subsets of S. If K is a non-empty subset of S, then the sum of elements of K is in the range $[0, n]$, because $0 \le x_i \le 1$. We divide the interval $[0, n]$ into $2^n - 1$ subintervals as follows:

$$[0, n] = \left[0, \frac{n}{2^n - 1}\right] \cup \left[\frac{n}{2^n - 1}, \frac{2n}{2^n - 1}\right] \cup ... \cup \left[\frac{(2^n - 2)n}{2^n - 1}, n\right] = I_1 \cup I_2 \cup ... \cup I_{2^n - 1}$$

Now, if the sum of the elements of some subset K of S is in the interval I_1, then we are done, because by setting $a_i = 1$ if $x_i \in K$ and 0 otherwise, we see that

$$\left|\sum_i a_i x_i\right| \le \frac{n}{2^n - 1}$$

So, suppose the sum of every subset K of S has sum in one of the intervals $I_2, ..., I_{2^n - 1}$. Since there are $2^n - 1$ non-empty subsets, by the pigeonhole principle it follows that two subsets K_1 and K_2 must have their sums in some interval I_j $(2 \le j \le 2^n - 1)$. Since the length of every interval is $\frac{n}{2^n - 1}$, it follows that

$$\left|\sum_{x \in K_1} x - \sum_{x \in K_2} x\right| \le \frac{n}{2^n - 1}$$

which implies that

$$\left|\sum_{x \in K_1 - K_2} x - \sum_{x \in K_2 - K_1} x\right| \le \frac{n}{2^n - 1}$$

Hence, if we set $a_i = 1$ if $x_i \in K_1 - K_2$, -1 if $x_i \in K_2 - K_1$ and 0 otherwise, we see that

$$\left|\sum_i a_i x_i\right| \le \frac{n}{2^n - 1}$$

and so the a_is have been found in all cases.

Now, we prove the general case. Let $x_1, ..., x_n$ be real numbers such that $|x_i| \le 1$ for each i. Let $y_i = |x_i|$. By what we proved above, there is a $b_i \in \{-1, 0, 1\}$ for each i such that

$$\left|\sum_i b_i y_i\right| \le \frac{n}{2^n - 1}$$

Now, we define $a_i = b_i$ if $x_i \ge 0$, and $a_i = -b_i$ otherwise. So, we have

$$\left|\sum_i a_i x_i\right| = \left|\sum_i b_i y_i\right| \le \frac{n}{2^n - 1}$$

(3). Consider the function $f_1 : \mathbb{N} \to \mathbb{N}$ given by

$$f(n) = 2n$$

for all $n \in \mathbb{N}$. This function is not onto, because the range does not contain any odd integer. However, it is one-one, because

$$2n_1 = 2n_2 \implies n_1 = n_2$$

Now, consider the function $f_2 : \mathbb{N} \to \mathbb{N}$ given by

$$f(n) = \left\lfloor \frac{n}{5} \right\rfloor + 1$$

Since n ranges over all naturals, the range of this map is $\mathbb{N}$, and hence it is onto. However, it is not one-one, because

$$f(1) = f(2) = 1$$

(4). Suppose (A, B) is an ordered pair such that $A \subseteq B \subseteq \{1, 2, ..., n\}$. Suppose B has k elements. Then, there are $\binom{n}{k}$ possible choices for B. Then, there are 2^k choices for A, because out of the k elements of B, we are constructing a subset of it. So, the total number of such pairs are

$$\#\text{ of pairs } (A, B) = \sum_{k=0}^{k} \binom{n}{k} 2^k = (1+2)^n = 3^n$$

(5). Let n be a natural number.

If $n = 1$, then it has an odd number of positive divisors, and evidently it is the square of itself. So, we will assume that $n > 1$.

Let $1 = a_1 < a_2 < ... < a_k = n$ be the list of the positive divisors of n. Given any divisor a_q of n, we know that $\dfrac{n}{a_q}$ is also a divisor of n. This way, we can pair up the divisors of n in such a way that a divisor occurs in exactly one (unordered) pair.

Now, suppose n has odd number of factors. If n was not a square number, then each pair $\left\{a_q, \dfrac{1}{a_q}\right\}$ would consist of distinct factors of n, and hence the total number of factors would be even, which is not possible. Hence, n must be a square number.

Conversely, if $n = d^2$, then any divisor a_q of n different from d can be paired up with another divisor $\dfrac{n}{a_q}$ of n. So, there are an even number of divisors of n other than d. So, it follows that n has odd number of divisors in total.

(6). Let us compute the number of monotone functions from $[n]$ to $[n]$.

We claim that the set of monotone functions is in bijection with the set of ordered pairs $(a_1, ..., a_n)$ where each a_i is non-negative, and

$$a_1 + ... + a_n = n$$

Given any integer solution to $a_1 + a_2 + ... + a_n = n$, there is a unique monotonic increasing function: map the first a_1 elements of $[n]$ to 1, the next a_2 elements to 2, and so on. Conversely, given any monotonic function, we check how many elements are mapped to $1, 2, ..., n$, and denote these numbers by $a_1, ..., a_n$. It is then easy to see that $a_1 + ... + a_n = n$. This establishes the bijection.

Now, as done in class, the number of non-negative integer solutions to

$$a_1 + ... + a_n = n$$

is equal to

$$\binom{n+n-1}{n-1} = \binom{2n-1}{n-1}$$

and hence this is equal to the number of monotonic increasing functions.

(7). Consider the expression $(2x + 7y^2 - 3z)^8$. We will compute the coefficient of $x^2y^8z^2$.

In the product $(2x+7y^2-3z^8)^8$, there are 8 identical terms. To make a term of the form $x^2y^8z^2$, we need to choose two $2x$ terms, four $7y^2$ terms and two $-3z$ terms from each product. Out of the 8 terms in the product, there are $\binom{8}{2}$ ways to choose two $2x$ terms. Out of the remaining 6 terms, there are $\binom{6}{4}$ ways to choose four $7y^2$ terms. Finally, there are $\binom{2}{2}$ ways to choose the remaining two $-3z$ terms. So, it follows that the coefficient of $x^2y^8z^2$ is

$$\binom{8}{2}\binom{6}{4}\binom{2}{2}2^27^4(-3)^2 = 36303120$$

(8). First, we partition $[2n]$ as follows

$$[2n] = \{1, 2n\} \cup \{2, 2n-1\} \cup ... \cup \{n, n+1\}$$

Observe that the sum of the two elements in the same part is $2n+1$. So, a subset of $[2n]$ which has no two elements adding up to $2n+1$ must contain atmost one element from each set in this partition.

So, for each set in the partition, we have two choices: pick one of the elements. So, it follows that there are 2^n subsets of $[2n]$ of size n such that there are no two elements in the subset whose sum is $2n+1$.

9. Here, we count the number of permutations of $[n]$ such that the sum of every consecutive pair of elements is odd. Such a permutation should have the following properties: first, no two even numbers can be placed next to each other, and second no two odd numbers can be placed next to each other. So, it follows that the odd numbers appear alternatively and the even numbers appear alternatively.

We deal with two cases: first, if n is an even number, then there are $\frac{n}{2}$ odd elements in $[n]$, and $\frac{n}{2}$ even elements in $[n]$. Now, the desired permutation can start with either in odd element or an even element. If it starts with an odd element, the total number of such permutations is $\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!$. Same is the case when the permutation starts with an even element. So, in all, there are $2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!$ such permutations.

Next, if n is odd, then the permutation must start with an odd number (because there are more odd numbers in $[n]$ than even numbers in this case). So, there are $\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!$ such permutations in this case.

(10). Let n be a positive integer. Here, we will count the total number of $n \times n$ matrices wit entries in $\{0, 1\}$ such that the row sum and column sum is even for every row and columns in the matrix. Let M be such a matrix, and let M_{ij} be the entries of the matrix.

First, let us introduce some notation. Let A_{n-1} denote the set of all $(n-1) \times (n-1)$ matrices X with entries in $\{0, 1\}$ such that the sum of all elements of X is even. Analogously, let A'_{n-1} be the set of all matrices with total sum odd.

We consider two cases:

(1) **Case 1:** Let B be the set of all matrices M having the given property such that $M_{11} = 0$ (i.e, the top-left element is 0). We claim that there is a bijection from B to A_{n-1}. So, let $M \in B$, and we map M to the unique sub-matrix obtained by removing the first row and the first column. Since the first row and column of M both have even sum and $M_{11} = 0$, it follows that the sum of the sub-matrix is even, and hence this sub-matrix is in A_{n-1}.

Conversely, given a matrix S in A_{n-1}, there is a matrix M in B such that S is the sub-matrix of M: we set $M_{11} = 0$, and above every column of S which has odd sum, we put a 1 in the first row of M, and above every column of S having even sum, we put a 0 in the first row of M. Similarly, for every row of S having an odd sum, we put a 1 in the first column of M, and for every row of S having even sum, we put a 0 in the first column of M. Because S can only have an even number of rows (and columns) having odd sum (because S is in A_{n-1}), it follows that all rows and columns of M have even sum, and hence $M \in B$. This establishes the bijection.

(2) **Case 2:** Let B' be the set of all matrices M having the given property such that $M_{11} = 1$ (i.e, the top-left element is 1). As in the first case, we claim that there is a bijection between B' and A'_{n-1}. So, suppose $M \in B'$, and we map M to the unique sub-matrix obtained after removing the first row and the first column of M. Since the first row and column of M have even sum and $M_{11} = 1$, it follows that the sum of the sub-matrix is odd, and hence the sub-matrix is in A'_{n-1}.

Conversely, given a matrix S in A'_{n-1}, there is a matrix M in B' such that S is the sub-matrix of M: we set $M_{11} = 1$, and above every column of S which has odd sum, we put a 1 in the first row of M, and above every column of S having even sum, we put a 0 in the first row of M. Similarly, for every row of S having an odd sum, we put a 1 in the first column of M, and for every row of S having even sum, we put a 0 in the first column of M. Because S can only have an odd number of rows (and columns) having odd sum (because S is in A'_{n-1}), it follows that all rows and columns of M have even sum, and hence $M \in B'$. This establishes the bijection.

So, it follows that we only need to count the number of elements in A_{n-1} and A'_{n-1}. However, observe that $A_{n-1} \cap A'_{n-1} = \phi$, and that

$$A_{n-1} \cup A'_{n-1} = P_{n-1}$$

where P_{n-1} is the set of all $(n-1) \times (n-1)$ matrices having entries in $\{0, 1\}$. So, it follows that

$$|A_{n-1} \cup A'_{n-1}| = |P_{n-1}| = 2^{(n-1)^2}$$

and hence this is the total number of required matrices.

2. Assignment 3

(1). First, the total number of compositions of 20 into 5 parts is

$$\# \text{ of compositions} = \binom{20-1}{5-1} = \binom{19}{4}$$

Next, the total number of compositions of 20 into 5 parts, where the second part is 1 is equal to the total number of compositions of 19 into 4 parts, which is

$$\binom{19-1}{4-1} = \binom{18}{3}$$

So, the total number of compositions of 20 into 5 parts with the second part not equal to 1 is

$$\binom{19}{4} - \binom{18}{3} = 3060$$

(2). We will show that

$$\sum_{k=1}^{n} (-1)^k k! S(n,k) = (-1)^n$$

where $S(n,k)$ is the Stirling number of the second kind.

Without loss of generality, suppose $n \in \mathbb{N}$ is even. The proof is the same when n is odd.

First, we note that for $1 \leq k \leq n$, the number $k!S(n,k)$ denotes the number of ways of partitioning the set $[n]$ into k parts, where the **ordering** of the partition matters. Let A_n be the number of ordered partitions of $[n]$ into even number of parts, and let B_n be the number of partitions of $[n]$ into odd number of parts.

We consider a map $P_n : S(A_n) \to B_n$ as follows (here $S(A_n)$ is a subset of A_n): for a partition in A_n, look at the element n: if it is a singleton part, then shift it to the previous part, otherwise make it into a singleton part (a singleton part is a set in the partition consisting of only one element). For instance, if $n = 5$, then the following two partitions are mapped as:

$$\{1,2\} \cup \{5\} \cup \{3\} \cup \{4\} \mapsto \{1,2,5\} \cup \{3\} \cup \{4\}$$
$$\{1,5\} \cup \{2\} \cup \{3\} \cup \{4\} \mapsto \{1\} \cup \{5\} \cup \{2,3\} \cup \{4\}$$

If the partition is of the form $\{n\} \cup Q$, then we map it to the partition $\{n\} \cup P_{n-1}(Q)$. It is clear by our construction that P_n is a one-one map. The only partition which does not have an image under P_n, i.e the only partition in A_n that is not in $S(A_n)$ is the partition

$$\{n\} \cup \{n-1\} \cup ... \cup \{1\}$$

and hence the cardinality of A_n is one more than the cardinality of B_n. So, we have

$$\sum_{k \text{ is even}} k!S(n,k) = 1 + \sum_{k \text{ is odd}} k!S(n,k)$$

and hence in this case we have

$$\sum_{k=1}^{n} (-1)^k k! S(n,k) = 1$$

and this completes the proof. (The proof of the case when n is odd is exactly the same)

Note: I discussed this problem with: Shibashis Mukhopadhyay, Pradyot Mohanty

(3). Let n be a positive integer. As proven in class, the total number of compositions of n is

$$\sum_{k=1}^{n}\binom{n-1}{k-1}=\sum_{k=0}^{n-1}\binom{n-1}{k}=2^{n-1}$$

The above expression is the binomial expansion of $(1+1)^{n-1}$. Now, we have

$$0=(1-1)^{n-1}=\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}$$

which implies that

$$\sum_{\substack{0\le k\le n-1\\ k \text{ is odd}}}\binom{n-1}{k}=\sum_{\substack{0\le k\le n-1\\ k \text{ is even}}}\binom{n-1}{k}$$

However, observe that

$$\sum_{\substack{0\le k\le n-1\\ k \text{ is odd}}}\binom{n-1}{k}=\sum_{\substack{1\le k\le n\\ k \text{ is even}}}\binom{n-1}{k-1}$$

and

$$\sum_{\substack{0\le k\le n-1\\ k \text{ is even}}}\binom{n-1}{k}=\sum_{\substack{1\le k\le n\\ k \text{ is odd}}}\binom{n-1}{k-1}$$

and so we get

$$\sum_{\substack{1\le k\le n\\ k \text{ is odd}}}\binom{n-1}{k-1}=\sum_{\substack{1\le k\le n\\ k \text{ is even}}}\binom{n-1}{k-1}$$

In the above equality, the left hand side counts the number of compositions of n into odd number of parts, and the right hand side counts the number of compositions of n into even number of parts. Since the total number of compositions is 2^{n-1}, it follows that the total number of compositions into even number of parts is 2^{n-2}.

(4). The problem corresponds to finding non-negative integers a_1, a_2, a_3, a_4, a_5 such that

$$a_1+a_2+a_3+a_4+a_5=10$$

and that $a_5<a_4$.

Define

$$A:=\{(a_1,...,a_5)|\ a_i\ge 0,\ \sum a_i=10,\ a_4>a_5\}$$
$$B:=\{(a_1,...,a_5)|\ a_i\ge 0,\ \sum a_i=10,\ a_4<a_5\}$$

and we are interested in the cardinality of A. We claim that A is in bijection with B. First, map $(a_1,...,a_5)\in A$ to the tuple $(a_1,...,a_5,a_4)$ (swapping the last two elements), which is evidently a tuple in B. Clearly, this is a one-one map. To show that it is onto, consider a tuple $x=(a_1,...,a_5)\in B$. Then it follows that $y=(a_1,...,a_5,a_4)\in A$, and y maps to x under the mapping. So, the mapping is a bijection.

Now, define the set

$$C:=\{(a_1,...,a_5)|\ a_i\ge 0,\ \sum a_i=10,\ a_4=a_5\}$$

It is clear that if $(a_1, ..., a_5) \in C$, then a_4 (and hence a_5) must be in the set $\{0, 1, 2, 3, 4, 5\}$. So, it follows that the cardinality of C is

$$\sum_{k=0}^{5} \binom{10-2k+2}{10-2k}$$

where the expression $\binom{10-2k+2}{2k}$ corresponds to the number of tuples (a_1, a_2, a_3) of non-negative integers such that

$$a_1 + a_2 + a_3 = 10 - 2k$$

The above sum is

$$\binom{12}{10} + \binom{10}{8} + \binom{8}{6} + \binom{6}{4} + \binom{4}{2} + \binom{2}{0} = 161$$

Now, $A \cup B \cup C$ is the set of all tuples $(a_1, ..., a_5)$ of non-negative integers such that

$$a_1 + a_2 + a_3 + a_4 + a_5 = 10$$

Also, A, B and C are pairwise disjoint, and as we have shown above, $|A| = |B|$. So, we have

$$2|A| + 161 = \binom{10+5-1}{5-1} = 1001$$

and hence

$$|A| = 420$$

(5). Let $n \in N$ be a positive integer. Here, we will find the sum of the first parts of all compositions of n. But first, we will prove a formula:

Lemma: For any positive integer n, we have

$$\sum_{k=1}^{n} k2^{-k} = 2^{-n}(-n + 2^{n+1} - 2)$$

Proof: For real $x \neq 1$, consider the geometric sum

$$\sum_{k=1}^{n} x^k = \frac{x(x^n - 1)}{x - 1}$$

Differentiating both sides, we see that

$$\sum_{k=1}^{n} kx^{k-1} = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$$

multiplying both sides by x, we see that

$$\sum_{k=1}^{n} kx^k = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$

Putting $x = \dfrac{1}{2}$, we see that

$$\sum_{k=1}^{n} k2^{-k} = 2^{-n}(-n + 2^{n+1} - 2)$$

Now, we do the problem. We know that for a positive integer n, the total number of compositions are 2^{n-1}. Observe that in any composition, the first part is in the set $\{1, 2, ..., n\}$. The number of compositions of n where the first part is k $(1 \leq k < n)$ is equal to the number of compositions of $n-k$. There is only one composition where the first part is n. So, it follows that the sum of the first parts of all compositions of n is

$$\begin{aligned}
\text{sum of first parts} &= n + \sum_{k=1}^{n-1} k(\#\text{ of compositions of } n-k) \\
&= n + \sum_{k=1}^{n-1} k2^{n-k-1} \\
&= n + 2^{n-1}\sum_{k=1}^{n-1} k2^{-k} \\
&= n + (1-n+2^n-2) \\
&= 2^n - 1
\end{aligned}$$

where we have used the previous **Lemma**.

So, it follows that the sum of first parts of all compositions of n is 2^n-1.

(6). Here, we will show that $S(n, n-m)$ is a polynomial of degree $2m$, where $S(n,k)$ denotes the Stirling number of the second kind.

Let's do so by induction on m. For the base case, we let $m = 0$. Then, for any $n \in \mathbb{N}$, we have

$$S(n, n-m) = S(n,n) = 1$$

which is evidently a polynomial of degree 0 in n. So, the base case has been proven.

Now, suppose the statement holds for some $m-1 \in \mathbb{N}$, i.e, $S(n, n-(m-1))$ is a polynomial of degree $2m-2$ in n, where $n \geq m$.

Now, consider $m \in \mathbb{N}$, and suppose $n \geq m+1$ is any integer. We have

$$S(n, n-m) = S(n-1, n-1-m) + (n-m)S(n-1, n-1-(m-1))$$

We now repeat the same process with the first term above, i.e we do

$$S(n-1, n-1-m) = S(n-2, n-2-m) + (n-1-m)S(n-2, n-2-(m-1))$$

We repeat the same process k times, where $n-k = m+1$. So, we get

$$\begin{aligned}
S(n, n-m) &= S(n-k, 1) + \sum_{k=0}^{n-m-1} (n-m-k)S(n-(k+1), n-(k+1)-(m-1)) \\
&= 1 + \sum_{k=0}^{n-m-1} (n-m-k)S(n-(k+1), n-(k+1)-(m-1))
\end{aligned}$$

Now, by our induction hypothesis, suppose

$$S(p, p-(m-1)) = a_{2m-2}p^{2m-2} + a_{2m-3}p^{2m-3} + \ldots + a_0 = P(p, m-1)$$

for all $p \geq m$, which is a degree $2(m-1)$ polynomial. So, we get

$$S(n, n-m) = 1 + \sum_{k=0}^{n-m-1} (n-m-k)P(n-(k+1), m-1)$$

For each k, the term

$$(n-m-k)P(n-(k+1), m-1)$$

is a degree $2m-1$ polynomial, and the leading term of this polynomial is $a_{2m-2}n^{2m-1}$. Since we are summing this leading term $n-m$ times, it follows that $S(n, n-m)$ is a polynomial whose leading term is $a_{2m-2}n^{2m}$, and hence it is a degree $2m$ polynomial. By induction, the proof is complete.

3. Assignment-4

(6). Consider the sequence

$$(d_1, d_2, d_3, d_4) = (3, 3, 1, 1)$$

Clearly, $d_1 + d_2 + d_3 + d_4$ is even, and each $d_i \leq 3$. However, we will show that there is no *simple* graph with this degree sequence.

By Havel-Hakimi theorem, $(3, 3, 1, 1)$ is graphic if and only if $(2, 0, 0)$ is graphic. However, it is easy to see that $(2, 0, 0)$ is *not* a graphic sequence, and hence $(3, 3, 1, 1)$ is also not graphic.

(7). Let G be a graph, and let A, B be two vertices.

Suppose there is a trail between A and B. So, the set of all trails between A and B is non-empty. Let p be the element of this set of minimum length. We claim that p is a path between A and B. To prove it, suppose p is not a path, i.e suppose some vertex is repeated. We represent the trail as

$$p = v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 ... v_i \xrightarrow{e_i} v_{i+1} ... v_j \xrightarrow{e_j} v_{j+1} ... v_{k-1} \xrightarrow{e_{k-1}} v_k$$

where each e_i is an edge, and $v_1 = A$ and $v_k = B$, and $v_i = v_j$ $(i \neq j)$ is the repeated vertex. Then, observe that the trail

$$p = v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 ... v_i \xrightarrow{e_j} v_{j+1} ... v_{k-1} \xrightarrow{e_{k-1}} v_k$$

is also a trail from A to B, but of *shorter* length, which is a contradiction. So, p must be a path.

In the above proof, we never actually used anything about trails. The same proof works if we replace the word "trail" by "walk".

8. Suppose each vertex in a simple graph has degree 4. Without loss of generality suppose the graph is connected. If it is not connected, then we will apply the following argument to every connected component, and hence we will be done.

Since G is connected and each vertex has degree 4, there is an Euler tour in the graph. Take a vertex $v_0 \in G$, and start the Euler tour from any edge of this vertex.

(9). In this problem, we determine the possible degrees and number of vertices in regular (simple) graphs which have a total of 22 edges. Let d be the common degree of each vertex in such a graph. We know that

$$\sum_{v \in V} d_v = 2|E|$$

and hence in this case we have

$$\sum_{v \in V} d = 44$$

which implies that

$$d|V| = 44$$

So, d must be in the set $\{1, 2, 4, 11, 22, 44\}$. Clearly, $d = 1$ and $|V| = 44$ is possible. Also, $d = 2$ and $|V| = 22$ is also possible. It is not hard to see that $d = 4$ and $|V| = 11$ is also possible. Finally, it is very easy to see that no other case is possible.

(10). Let G be a simple graph such that every vertex has degree k. We will show that G contains a cycle of length greater than or equal to $k+1$.

4. Partially Ordered Sets and Mobius Inversion

A *partially ordered set* (or poset) P is a set with a relation $\leq$ which is reflexive, anti-symmetric and transitive.

We say that two posets P and Q are *isomorphic* if there is an order preserving bijection between them. This bijection is also called the *isomorphism* between the two posets.

Next, we introduce the notion of an *incidence algebra* on a poset. For a poset P, we define the *incidence algebra* $I(P)$ as

$$I(P) := \{f : P \times P \to P |\ f(x,y) = 0 \text{ if } x \not\leq y\}$$

For instance, the zero function is in $I(P)$, the *Krocker-Delta* (δ) function is in $I(P)$. Consider the zeta function ζ defined as $\zeta(x,y) = 1$ if $x \leq y$, and 0 otherwise. It is clear that $\zeta \in I(P)$. It is not hard to see that $I(P)$ forms a vector space over $\mathbb{R}$.

We now make $I(P)$ into a ring by introducing the *convolution* of two functions. For $f, g \in I(P)$, define the *Dirichlet convolution* as

$$(f * g)(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y)$$

Let us prove some properties of convolutions:

Theorem 4.1. Let $f, g, h \in I(P)$ for some poset P. Then:
(1) $(f * g) * h = f * (g * h)$
(2) $f * \delta = \delta * f = f$, where δ is the Kronecker function.

Proof: To prove (1), let $x, y \in P$. Then,

$$\begin{aligned}
(f * g) * h(x,y) &= \sum_{x \leq z \leq y} (f * g)(x,z)h(z,y) \\
&= \sum_{x \leq z \leq y} \left(\sum_{x \leq z_1 \leq z} f(x,z_1)g(z_1,z) \right) h(z,y) \\
&= \sum_{x \leq z \leq y} \left(\sum_{x \leq z_1 \leq y} f(x,z_1)g(z_1,z) \right) h(z,y) \\
&= \sum_{x \leq z \leq y} \sum_{x \leq z_1 \leq y} f(x,z_1)g(z_1,z)h(z,y) \\
&= \sum_{x \leq z_1 \leq y} f(x,z_1) \sum_{x \leq z \leq y} g(z_1,z)h(z,y) \\
&= \sum_{x \leq z_1 \leq y} f(x,z_1) \sum_{z_1 \leq z \leq y} g(z_1,z)h(z,y) \\
&= \sum_{x \leq z_1 \leq y} f(x,z_1)(g * h)(z_1,y) \\
&= f * (g * h)
\end{aligned}$$

We now prove (2). Observe that

$$f * \delta(x,y) = \sum_{x \le z \le y} f(x,z)\delta(z,y) = f(x,y)$$

and hence we are done.

We now see when functions in $I(P)$ are invertible w.r.t the convolution:

Theorem 4.2. $f \in I(P)$ is invertible if and only if $f(x,x) \neq 0$ for all $x \in P$.

Proof: First, suppose f is invertible. Then,

$$f * f^{-1}(x,x) = \sum_{x \le z \le x} f(x,z)f^{-1}(z,x) = \delta(x,x) = 1$$

and hence

$$f(x,x)f^{-1}(x,x) = 1$$

which implies that $f(x,x) \neq 0$. Infact, this gives us the formula

$$f^{-1}(x,x) = \frac{1}{f(x,x)}$$

Now, suppose $f(x,x) \neq 0$ for all $x \in P$. We construct f^{-1} as follows: set

$$f^{-1}(x,x) = \frac{1}{f(x,x)}$$

Take any $x, y \in P$ such that $x \neq y$. We want

$$f^{-1} * f(x,y) = \sum_{x \le z \le y} f^{-1}(x,z)f(z,y) = \delta(x,y) = 0$$

and so we get

$$\sum_{x \le z < y} f^{-1}(x,z)f(z,y) + f^{-1}(x,y)f(y,y)$$

and hence

$$f^{-1}(x,y) = \frac{-1}{f(y,y)} \sum_{x \le z < y} f^{-1}(x,z)f(z,y)$$

So, f^{-1} can be computed by induction.

Note: Note that, by our construction, we have only shown that f^{-1} is the *left* inverse of f. We need not check that f^{-1} is also the *right* inverse, because in any associative operator that has left inverses, left and right inverses are the same.

From the previous theorem, it follows that the ζ function is invertible, for any poset. We denote its inverse by μ, and call it the *Mobius* function.

The Mobius function has a very important application which we now prove, called *Mobius inversion*:

Theorem 4.3. Suppose P is any poset, and let $f : P \to \mathbb{R}$ be any function. Suppose p has a minimal element, say m. Then, put

$$g(y) = \sum_{x \le y} f(x)$$

so that $g : P \to \mathbb{R}$. Then,

$$f(y) = \sum_{x \le y} \mu(x,y)g(x)$$

Proof: Let m be the minimal element. First, we extend f to $P \times P$ as follows: if for any $y \in P$, put

$$f'(m, y) = f(y)$$

and for any $(x, y) \in P$ such that $x \neq m$, put

$$f'(x, y) = 0$$

Clearly, we can see that $f' \in I(P)$. Similarly, we extend g to $P \times P$ and call it g'.

We claim that

$$g' = f' * \zeta$$

To prove this, first suppose $y \in P$. Then,

$$g'(m, y) = g(y) = \sum_{x \leq y} f(x) = \sum_{m \leq x \leq y} f'(m, x)\zeta(x, y)$$

and if $x \neq m$, we have

$$g'(x, y) = 0 = \sum_{x \leq z \leq y} f'(x, z)\zeta(z, y)$$

and the claim is true. So, we have

$$f' = g' * \mu$$

and so for any $y \in P$, we have

$$f(y) = f'(m, y) = \sum_{m \leq z \leq y} g'(m, z)\mu(z, y) = \sum_{x \leq y} g(x)\mu(x, y)$$

and the proof is complete.

Mobius function for the Boolean algebra: Suppose P is a boolean algebra on n elements. Let $X, Y \in P$. Then, we claim

$$\mu(X, Y) = (-1)^{|Y - X|}$$

Proof: First, it is easy to see that

$$\mu(X, Y) = \mu(\phi, Y - X)$$

Now, observe that

$$\mu(\phi, \phi) = 1$$

Suppose the statement is true for all Y and X such that $|Y - X| \leq k$ for some positive integer k.

Mobius function for the divisor lattice: Here, we have

$$\mu(d, n) = \mu\left(1, \frac{n}{d}\right)$$

We know that

$$n = \sum_{d|n} \phi(d)$$

and applying Mobius inversion, we get

$$\phi(n) = \sum_{d|n} d\mu(d, n) = \sum_{d|n} d\mu\left(\frac{n}{d}\right)$$

where μ is the Mobius function which we defined for natural numbers.

Inclusion-Exclusion: Here we prove the IEP using Mobius inversion. The proof is as follows: Let $I \subseteq \{1, 2, ..., m\}$

$$f(I) = |\{x|\ x \text{ occurs in exactly in } A_j, j \in I\}|$$

and let

$$g(I) = |\{x|\ x \text{ is present in } A_i, i \in I\}|$$

So, we have that

$$g(J) = \sum_{J \subseteq I} f(I)$$

We can again apply Mobius inversion above, except the relation has reversed. So, we get

$$f(J) = \sum_{J \subseteq I} \mu(J, I) g(I)$$

Now, observe that

$$f(\phi) = |X / A_1 \cup A_2 \cup ... \cup A_n|$$

and hence we get

$$\begin{aligned} f(\phi) &= \sum_{\phi \subseteq I} \mu(\phi, I) g(I) \\ &= \sum_{\phi \subseteq I} (-1)^{|I|} g(I) \\ &= \sum_{\phi \subseteq I} (-1)^{|I|} g(I) \end{aligned}$$

which gives us the inclusion-exclusion principle.

5. Graph Theory

A graph is a pair $G = (V, E)$ where $E \subseteq V \times V$. For us, there are no self loops.

The *neighborhood* of a vertex is the set of vertices to which the vertex is directly connected, and the vertex itself. The *degree* of a vertex is the number of edges incident to that vertex. We use the notation $\Delta(G)$ to denote the largest degree in the graph, and similarly $\delta(G)$ is the smallest degree in the graph. A vertex is called *isolated* if its degree is zero.

Here are some interesting properties of graphs:

Theorem 5.1. For any graph $G = (V, E)$, we have

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof: It is a very easy proof.

Now, we will discuss the *degree sequence* of a graph. Given a graph $G = (V, E)$, the *degree sequence* is the sequence

$$d_1 \geq d_2 \geq ... \geq d_{|V|}$$

where each d_i is the degree of some vertex of the graph, and we arrange these in descending order. A sequence $(d_1, ..., d_n)$ of non-negative integers is said to be a *graphic sequence* if it is the degree of some graph on n vertices. Let us now prove a simple fact about degree sequences, which will enable us to find a way of checking whether a sequence is a graphic sequence or not:

Theorem 5.2. Let $(d_1, ..., d_n)$ be a non-increasing sequence of non-negative integers. Then, it is a graphic sequence if and only if there is a simple graph G on n vertices having the same degree sequence, such that vertex v_1 is connected to vertices $v_2, v_3, ..., v_{d_1+1}$.

Proof: One direction of the statement is clear. So, let's prove the other direction. Suppose $(d_1, .., d_n)$ is a graphic sequence. Let G be the graph in which vertex v_1 is connected to the maximum number of vertices in the set $\{v_2, ..., v_{d_1+1}\}$ (such a maximal graph can be found since there are only finite number of graphs). We claim that in this graph, v_1 must be connected to all vertices in the set $\{v_1, ..., v_{d_1+1}\}$. Suppose this is not true. Let s be a vertex in this set to which v_1 is not connected, and let w be a vertex outside this set to which v_1 is connected (w must exist because $\deg(v_1) = d_1$). Now, there must be some vertex q in the graph to which s is connected, and w is not connected (this is true because $\deg(s) \geq \deg(w)$). We now make a new graph: remove the edges (v_1, w) and (s, q), and add the edges (v_1, s) and (w, q). Note that the degrees of all vertices are preserved, and we have generated a new graph, in which v_1 is connected to more vertices in the set $\{v_2, ..., v_{d_1+1}\}$, which contradicts the fact that G is maximal. This completes the proof.

Using this fact, we will now prove the **Havel-Hakmimi algorithm** to determine whether a sequence is graphic or not:

Theorem 5.3. Havel-Hakimi: A sequence $(d_1, ..., d_n)$ is graphic if and only if the sequence $(d_2 - 1, ..., d_{d_1+1} - 1, ...d_n)$ is a graphic sequence.

Proof: This is straightforward. Applying the previous theorem, we just delete the first vertex and all its edges, and consider the new graph.

Now let us define some new terms. A *trail* is a sequence of distinct edges, such that consecutive edges share a common vertex. A *closed trail* is a trail where the starting vertex and the ending vertex are the same. A *walk* is a sequence of edges which are not necessarily distinct. A *closed walk* is analogously defined. A *path* is a trail in which no vertex is repeated. Two vertices u and v are said to be *connected* if there is a path between them.

In the next theorem, we will see when a connected graph has an *Euler tour*: an *Euler tour* is a closed trail which consists of all edges of the graph.

Theorem 5.4. A connected graph G has an Euler tour if and only if every vertex has odd degree.

Proof: First, suppose that a connected graph G has an Euler tour. Let v_0 be the starting vertex. Let us show that v_0 has even degree. To see this, observe that every time the trail leaves v_0, it must enter it as well. So, we can pair up edges incident on v_0, and hence the degree of v_0 is even. Now, suppose v is a vertex that is not the starting vertex. If the degree of v is zero, then there is nothing to prove. If the degree is non-zero, then the Euler tour must visit all edges incident on v. So, every time the tour enters v, it must exit it as well, and hence the degree of v is even.

Conversely, suppose all vertices have even degree. Pick any vertex v_0 in the graph.

We similarly define the notion of an *Euler trail.* An *Euler trail* is a trail starting at a vertex and ending at another distinct vertex, such that all edges are visited in the trail. We can prove a simple criteria for determining when a graph has an Euler trail:

Theorem 5.5. Let G be a finite connected graph. Then, G has an Euler trail if and only if there are exactly two vertices of odd degree.

Proof: Let u, v be the distinct vertices of odd degree. We make a new edge (u, v). Then, observe that the resultant graph has only even degree vertices, and hence there is an Euler tour in the resultant graph. Start this tour from the edge (u, v). Then, removing this edge from the tour, we obtain a trail starting at v and ending at u and visiting all edges.

Conversely, suppose G has an Euler trail. Suppose the trail starts at a vertex u and ends at a vertex v, both of which are distinct. So, it must be true that the degrees of u and v are even. For all other vertices, the trail enters and leaves the vertices equally often, and hence all other vertices have even degree.

6. Trees

A *tree* is a *minimally* connected simple graph. Another definition can be a connected acylic graph, which is guaranteed by the following easy to prove theorem:

Theorem 6.1. Let G be a connected simple graph. Then, G is minimally connected if and only if G is acylic.

Yet another definition could be the following: a simple graph G in which between any two vertices there is exactly one path is called a tree. It is also easy to show that a connected simple graph is a tree if and only if the number of edges in it is $|V| - 1$.

The next result is highly non-trivial:

Theorem 6.2. The number of trees on $[n]$ vertices is n^{n-2}.

We will see this after proving the *matrix tree theorem.* However, I would recommend the reader to refer to the proof of this theorem given by *André Joyal.*

From this, it is easy to see that the number of rooted trees on n vertices is n^{n-1}, and the number of rooted *forests* (a *forest* is just an acylic graph) on n vertices is

7. Spectral Graph Theory

In this section, we will study graph theory algebraically, i.e studying graphs using *matrices* associated with them.

Consider the adjacency matrix A_G of a directed (or undirected) graph. The following is a fundamental result:

Theorem 7.1. Let i, j be vertices of G. Then, the number A^k_{ij} is the number of walks of length k between the vertices i and j.

Proof: It is a simple proof by induction. Try doing it.

A_G gives some simple tests regarding graph connectivity. The first one is the following:

Theorem 7.2. Let G be a simple graph on n vertices, and let A_G be the adjacency matrix. Then, G is connected if and only if $(I + A)^{n-1}$ consists of strictly positive entries.

Proof: First, consider the following elementary fact:

$$(I + A)^{n-1} = \sum_{k=0}^{n-1} \binom{n}{k} A^k$$

Let i, j be any two vertices. If G is connected, then for some k, A^k_{ij} is non-zero (because the maximum path length is $n - 1$), and hence all entries of $(I + A)^{n-1}$ are strictly positive. Conversely, if all entries are strictly positive, there must be some $0 \leq k \leq n-1$ for which A^k_{ij} is positive, and hence there is a walk from i to j, meaning that there is a path. The claim follows.

We now introduce the *Laplacian* of a simple graph. It is defined as

$$L_G = D_G - A_G$$

where D_G is the diagonal matrix consisting of the degrees of the vertices, and A_G is the usual adjacency matrix.

8. Assignment-5

(1). Let G be a loopless undirected graph. We will show that $L(G)$ (Laplacian) is a sum of $|E|$ positive semi-definite matrices of rank 1.

Suppose $(u, v) \in E$, where $u, v \in \{1, 2, ..., n\}$, where the vertices are labelled with numbers $\{1, 2, ..., n\}$. Define a matrix $B_{(u,v)}$ by the rule

$$B_{(u,v)}(ij) = \begin{cases} 1, & i = j = u \text{ or } i = j = v \\ -1, & i = u, j = v \text{ or } i = v, j = u \\ 0, & \text{otherwise} \end{cases}$$

We claim that

$$L(G) = \sum_{(u,v)\in E} B_{(u,v)}$$

First, consider a diagonal entry $L(G)(ii)$. The sum on the right hand side contributes $\deg(i)$ to the diagonal entry, because each edge of the vertex i is considered exactly once. Now, let $L(G)(ij)$ be an off diagonal entry. If (i, j) is an edge, the sum on the right hand side will contribute -1 to both the off diagonal entries $L(G)(ij)$ and $L(G)(ji)$. If (i, j) is not an edge, then the right hand side will contribute nothing to the off diagonal entries. This shows the equality.

Now, the second claim is that $B_{(u,v)}$ is a positive semi-definite matrix for each edge (u, v). Without loss of generality suppose $u < v$. Let $x = (x_1, ..., x_u, ..., x_v, ..., x_n)$ be any vector in $\mathbb{R}^n$. Then, we have

$$B_{(u,v)}x = \begin{bmatrix} 0 \\ 0 \\ ... \\ x_u - x_v \\ ... \\ x_v - x_u \\ ... \\ 0 \end{bmatrix}$$

where $x_u - x_v$ is the u^{th} row, and $x_v - x_u$ is the v^{th} row, which follows by the definition of $B_{(u,v)}$ and matrix multiplication. Hence, we have

$$x^T B_{(u,v)}x = x_u(x_u - x_v) + x_v(x_v - x_u) = (x_u - x_v)^2 \geq 0$$

and hence $B_{(u,v)}$ is positive semi-definite. Since $L(G)$ is a sum of positive semi-definite matrices, it follows that $L(G)$ is also positive semi-definite.

(2). Let G be an undirected simple graph. We will first prove the following lemma:

Lemma 1: If G has k ($k > 1$) connected components, then the multiplicity of the 0 eigenvalue of $L(G)$ is exactly k. If G is connected, then the multplicity of the 0 eigenvalue of $L(G)$ is exactly 1.

Proof: First, we prove the second part of the theorem, and let G be connected. We know that $L(G)$ is of the form

$$L(G) = \begin{bmatrix} d_1 & ... & ... & ... \\ ... & d_2 & ... & ... \\ ... & ... & ... & ... \\ ... & ... & ... & d_n \end{bmatrix}$$

where $\{d_1, ..., d_n\}$ are the degrees of the vertices $\{1, ..., n\}$. Also, we know that $(1, 1.., 1)$ (n times) is an eigenvector with eigenvalue 0.

Now, suppose vertex 1 is connected to the vertices $\{i_{11}, i_{12}, ..., i_{1d_1}\}$ (there will be d_1 vertices in this list). In general, suppose vertex k is connected to vertices $\{i_{k1}, i_{k2}, ..., i_{kd_k}\}$ where $1 \leq k \leq n$. Now, suppose $x = (x_1, ..., x_n) \in \mathbb{R}^n$ is an eigenvector with eigenvalue 0. So, we know that

$$L(G)x = 0$$

and hence by multipliying the two matrices explicitly, we get that

$$L(G)x = \begin{bmatrix} d_1x_1 - x_{i_{11}} - ... - x_{i_{1d_1}} \\ d_2x_2 - x_{i_{21}} - ... - x_{i_{2d_2}} \\ ... \\ d_nx_n - x_{i_{n1}} - ... - x_{i_{nd_n}} \end{bmatrix} = 0$$

and hence we get the following equations:

$$\begin{aligned} d_1x_1 &= x_{i_{11}} + ... + x_{i_{1d_1}} \\ d_2x_2 &= x_{i_{21}} + ... + x_{i_{2d_2}} \\ ... &= ... \\ d_nx_n &= x_{i_{n1}} + ... + x_{i_{nd_n}} \end{aligned}$$

Now, without loss of generality suppose $\max\{x_1, ..., x_n\} = x_n$. Then, the last equation above implies that

$$x_n = x_{i_{n1}} = ... = x_{i_{nd_n}}$$

because x_n is greater than or equal to each of $x_{i_{nj}}$, and d_n is a positive integer.

Now, we repeat the same argument with the coordinates $x_{i_{n1}}$ till $x_{i_{nd_n}}$. Since G is connected, this finite process will show that $x_1 = x_2 = ... = x_n$. And hence

$$x = x_1(1, 1, ..., 1)$$

so that the (geometric) multiplicity of the 0 eigenvalue is 1. Since $L(G)$ is diagonalisable, it follows that the algebraic multiplicity of 0 is also 1.

We now prove the first part of the theorem. Suppose G has k connected components. Let the k connected components be $C_1, ..., C_k$. If necessary, relabel the vertices so that the first C_1 labels are in C_1, then next C_2 labels are in C_2 and so on. In that case, the

Laplacian $L(G)$ is a block diagonal matrix of the form

$$L(G) = \begin{bmatrix} L_1 & 0 & ... & 0 \\ 0 & L_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & 0 & L_n \end{bmatrix}$$

where L_i is the Laplacian matrix of the i^{th} connected component. Now, the characteristic polynomial of L is

$$\det(L - xI) = \det(L_1 - xI_1) ... \det(L_k - xI_k)$$

where the last equation is true because of the diagonal blocks of L, and I_j is the identity square matrix of size $|C_j|$. Since each L_j represents a connected graph, it follows that the multiplicity of 0 in the polynomial $\det(L_j - xI_j)$ is exactly one (by what we proved above), and hence the multiplicity of 0 in $\det(L - xI)$ is exactly k. Since L is diagonalisable (being real symmetric), it follows that the algebraic and geometric multiplicities of 0 are both k.

The above lemma proves that: if G has k connected components (where $k \geq 1$), then the multiplicity of the 0 eigenvalue of $L(G)$ is exactly k. Note that this also proves that if the multiplicity is k, then G must have k-connected components.

We now give a set of linearly independent eigenvectors. If G is connected, the set is $(1, 1, ..., 1)$ (n times). If G has k connected components which are denoted by $C_1, ..., C_k$, where $k > 1$, then as before let the matrix $L(G)$ be of the form

$$L(G) = \begin{bmatrix} L_1 & 0 & ... & 0 \\ 0 & L_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & 0 & L_k \end{bmatrix}$$

where each L_i is the Laplacian of the i^{th} connected components. Let v_1 be the vector whose first $|C_1|$ coordinates are 1 and the rest 0. Let v_2 be the vector whose next $|C_2|$ coordinates are 1 and the rest 0. Similarly, let v_i be the vector whose coordinates are all 0, except those coordinates which are indexed by the vertices in C_i. It is clear that these are k linearly independent vectors. Also, observe that these are actually eigenvectors with eigenvalue 0, because the coordinates of Lv_i indexed by vertices in C_i are just the coordinates of $L_i(1, 1, ..., 1)$ (here 1 occurs $|C_i|$ times), and $L_i(1, ..., 1) = 0$. So, this is the set of linearly independent eigenvectors.

3. Suppose G is the star graph on n vertices, and without loss of generality suppose vertex 1 has degree $n - 1$, and all other vertices have degree 1. Then, the adjacency matrix of G is of the form

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & ... & 1 \\ 1 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 1 & 0 & 0 & ... & 0 \end{bmatrix}$$

Since $A(G)$ is real and symmetric, all its eigenvalues are real. So, suppose λ is an eigenvalue, and suppose $x = (x_1, ..., x_n) \neq O$ is an eigenvector. Then, we have

$$A(G)x = \lambda x$$

and by multiplying the matrices, we get that

$$\begin{bmatrix} 0 & 1 & 1 & ... & 1 \\ 1 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 1 & 0 & 0 & ... & 0 \end{bmatrix} x = \begin{bmatrix} x_2 + ... + x_n \\ x_1 \\ x_1 \\ ... \\ x_1 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ ... \\ x_n \end{bmatrix}$$

and hence we get the system of equations

$$\begin{aligned} \lambda x_1 &= x_2 + ... + x_n \\ x_1 &= \lambda x_2 \\ x_1 &= \lambda x_3 \\ ... &= ... \\ x_1 &= \lambda x_n \end{aligned}$$

Now, if $\lambda = 0$, then $x_1 = 0$, and $x_2 + ... + x_n = 0$. So, we can find $n-2$ linearly independent eigenvectors with eigenvalue 0, and hence the geometric (as well as algebraic, since $A(G)$ is diagonalisable) multiplicity of 0 is $n-2$. If $\lambda \neq 0$, then we obtain

$$x_2 = x_3 = ... = x_n = \frac{x_1}{\lambda}$$

and hence

$$\lambda x_1 = (n-1)\frac{x_1}{\lambda}$$

and hence (since $x_1 \neq 0$ because x is an eigenvector)

$$\lambda^2 = (n-1)$$

and hence the non-zero eigenvalues are $\pm\sqrt{n-1}$.

4. Let G be a cycle of length n. Then, the graph G looks like

$$...—n—1—2—3—...$$

and hence the adjacency matrix is

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & ... & 1 \\ 1 & 0 & 1 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 1 & 0 & 0 & ... & 0 \end{bmatrix}$$

So, every row of $A(G)$ has exactly two 1s, and the same is true for every column as well. Now, consider the two permutations

$$1 \to 2 \to 3 \to ... \to n-1 \to n \to 1$$

and

$$n \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow ... \leftarrow n-1 ... \leftarrow n$$

which are clearly inverses of each other. Let P be the matrix representing the first permutation, and let P^{-1} be the matrix representing the second permutation. Then, $P(i, i+1) = 1$ for each $1 \leq i \leq n-1$, and $P(n,1) = 1$, and all other entries are 0.

Similarly, $P^{-1}(i, i-1) = 1$ for each $2 \leq i \leq n$, and $P^{-1}(1, n) = 1$, and all other entries are 0. It is then easy to see that

$$A(G) = P + P^{-1}$$

and hence $A(G)$ is a sum of a permutation matrix and its inverse.

We first make a remark: observe that the degree matrix $D(G)$ is nothing but $2I_n$, because G is a 2-regular graph. So, the eigenvalues of $L(G)$ are translates of the eigenvalues of $A(G)$: specifically, eigenvalues of $L(G)$ are of the form $2 - \lambda$, where λ is an eigenvalue of $A(G)$. Also, the multiplicities of the eigenvalues are the same as those of the multiplicities in $A(G)$. So, it suffices to work with $A(G)$.

So, we will now compute the eigenvalues of $A(G)$ using the permutation matrix P.

Now, suppose λ is an eigenvalue of P. Clearly, $\lambda \neq 0$ because P is invertible, and hence λ^{-1} is an eigenvalue of P^{-1} (the eigenvector being the same). So, it follows that eigenvalues of A are of the form $\lambda + \lambda^{-1}$, where λ is an eigenvalue of P.

Now, if λ is an eigenvalue of P (for now we allow λ to be complex as well), and if $x = (x_1, ..., x_n)$ is an eigenvector, then we have

$$Px = \lambda x$$

which implies that

$$(x_2, ..., x_n, x_1) = \lambda(x_1, ..., x_n)$$

which implies that

$$x_1 = \lambda x_1 = ... = \lambda^n x_1$$

and hence $x_1 \neq 0$ (because x is an eigenvector), and hence we have

$$\lambda_n = 1$$

which implies that λ is an n^{th} root of unity. So, the spectrum of P is the set of n^{th} roots of unity, which are

$$\lambda = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

for $k = 0, ..., n-1$. So, we have

$$\lambda + \lambda^{-1} = 2\cos\left(\frac{2\pi k}{n}\right)$$

for $k = 0, ..., n-1$, and hence these are the eigenvalues of $A(G)$. Note that since $A(G)$ is diagonalisable, all its eigenvalues are real, and since we have found n distinct eigenvalues, it follows that the geometric (as well as algebraic, because $A(G)$ is diagonaliable) multiplicity of each eigenvalue is 1.

Now, the eigenvalues of $L(G)$ are

$$\lambda = 2\left(1 - \cos\left(\frac{2\pi k}{n}\right)\right)$$

for $k = 0, ..., n-1$. By the matrix tree theorem, we know that the number of spanning trees of G the product of the non-zero eigenvalues divided by n, and hence

$$\#\text{of spanning trees} = \frac{1}{n}\prod_{k=1}^{n-1} 2\left(1 - \cos\left(\frac{2\pi k}{n}\right)\right)$$

Now, it is easy to see that the number of spanning trees of the cycle graph is n, because removing any edge gives a spanning tree. So, it follows that

$$\frac{1}{n}\prod_{k=1}^{n-1} 2\left(1-\cos\left(\frac{2\pi k}{n}\right)\right) = n$$

which is the required trigonometric identity.

5. Let G be a d-regular graph on n vertices and consider the matrix $M = \dfrac{1}{d}A(G)$.

First, if $x = (x_1, ..., x_n) \in \mathbb{R}^n$ is any vector, then we have

$$x^T Mx = \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}x_i x_j$$

which is clear by the definition of inner products.

Next, we have

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}(x_i+x_j)^2 &= \sum_{i=1}^{n}\sum M_{ij}x_i^2 + M_{ij}x_j^2 + 2M_{ij}x_i x_j \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}x_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{} M_{ij}x_j^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}x_i j_j \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}x_i^2 + \sum_{j=1}^{n}\sum_{i=1}^{n} M_{ji}x_j^2 + 2x^t Mx \\
&= 2\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}x_i^2 + 2x^t Mx
\end{aligned}$$

where in the second last step we used the fact that M is symmetric, and that order of summation can be changed in finite sums. Continuing this, we get

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}(x_i+x_j)^2 &= 2\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}x_i^2 + 2x^t Mx \\
&= 2\sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} M_{ij} + 2x^t Mx \\
&= 2\sum_{i=1}^{n} x_i^2 + 2x^t Mx \\
&= 2x^t x + 2x^t Mx \\
&= 2(x^t x + x^t Mx)
\end{aligned}$$

where we used the fact that the sum of a row in M is equal to 1, because the graph is d-regular.

Now, observe that the sum

$$\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}(x_i+x_j)^2$$

is non-negative for all $x \in \mathbb{R}^n$. So, it follows that

$$2(x^t x + x^t Mx) \geq 0$$

for all unit vectors $x \in \mathbb{R}^n$. If λ_n is the minimum eigenvalue of M, then we have (by the formula $\lambda_n = \inf x^t Mx$ for unit vectors x)

$$2(1+\lambda_n) \geq 0$$

which implies that

$$\lambda_n \geq -1$$

Now, we will prove that $\lambda_n = -1$ if and only if G has a bipartite component. Without loss of generality, we assume that G is connected(the proof when G is not connected is given after this).

First, suppose G is bipartite. We know that a graph is bipartite if and only if its spectrum is symmetric. Observe that 0 is always an eigenvalue of the Laplacian of G, and hence d is always an eigenvalue of $A(G)$, which means that 1 is always an eigenvalue of M. So, it follows that -1 is also an eigenvalue, and hence $\lambda_n = -1$.

Conversely, suppose $\lambda_n = -1$, and let $x = (x_1, ..., x_n)$ be an eigenvector. So, it follows that

$$2(x^t x + x^t Mx) = 2(x^t x - x^t x) = 0$$

which implies that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}(x_i + x_j)^2 = 0$$

which implies that $M_{ij}(x_i + x_j)^2 = 0$ for each $1 \leq i, j \leq n$, and hence if ij is an edge, then $x_i = -x_j$. For the sake of contradiction, suppose G is not bipartite, i.e it has an odd cycle. Let $i_1 i_2 ... i_n$ be the odd cycle, where $i_n = i_1$. Because n is odd (odd cycle), it follows that

$$x_{i_1} = -x_{i_2} = x_{i_3} = ... = x_{i_n} = -x_{i_1}$$

and hence $x_{i_1} = x_{i_2} = ... = x_{i_n}$. Since G is connected, this implies that $x_1 = x_2 = ... = x_n = 0$, which means that $x = 0$. But, this is a contradiction, because x was assumed to be an eigenvector. Hence, G must be bipartite, and we are done.

Now if G is disconnected, suppose $C_1, ..., C_k$ are the connected components, and we write the matrix M as

$$M = \begin{bmatrix} M_1 & 0 & 0 & ... & 0 \\ 0 & M_2 & 0 & ... & 0 \\ ... & ... & ... & ... & \\ 0 & 0 & 0 & ... & M_k \end{bmatrix}$$

where each M_i is the corresponding matrix for the i^{th} connected component. So, if G has a bipartite component, -1 will be an eigenvalue for some M_i (for some connected component), and hence it will be an eigenvalue for M (we can just put zeroes at coordinates which are not indexed by the vertices in C_i to get an eigenvector). Conversely, if -1 is an eigenvalue of M, it is an eigenvalue for some M_i, and in that case the component C_i will be bipartite. So the statement is true for G.

9. Assignment-6

1. Let A be a square matrix with non-zero entries such that the sum of every row and every column of the matrix is r, where $r > 0$ is an integer. We will show that A can be written as a sum of permutation matrices. We will do so by induction on r.

First, if $r = 1$, then for every row and every column has exactly one 1, implying that A is a permutation matrix. So the base case is done.

For the inductive case, suppose the statement is true for some $r \in \mathbb{N}$, and let A be a matrix such that the row and column sum is $r+1$. We will show that we can deduct a permutation matrix from A, i.e there is some permutation matrix P such that $A - P$ has column sum and row sum equal to r. Then, by applying the induction to the matrix $A - P$, the claim will be proven.

So, consider the following graph: let $X = Y = \{1, ..., n\}$ and consider the bipartite graph with bipartition (X, Y), where there is an edge between i and j if $A_{ij} > 0$. We show that this graph has a perfect matching, i.e there is some permutation σ such that $A_{i,\sigma(i)} > 0$ for each i, and then the permutation matrix to delete from A will be the matrix P given by

$$P_{ij} = \begin{cases} 1, & \text{if } j = \sigma(i) \\ 0, & \text{otherwise} \end{cases}$$

To show that the graph has a perfect matching, we will prove that the graph satisfies Hall's condition. Let $I \subset X$. Let $i \in I$. Then observe that

$$\sum_{j \sim i} A_{ij} = r + 1$$

where $j \sim i$ means that there is an edge between i and j. So, we see that

$$\begin{aligned} |I| &= \frac{1}{r+1} \sum_{i \in I} \sum_{j \in N(I)} A_{ij} \\ &= \frac{1}{r+1} \sum_{j \in N(I)} \sum_{i \in I} A_{ij} \\ &\leq \frac{1}{r+1} \sum_{j \in N(I)} \sum_{i \in X} A_{ij} \\ &= |N(I)| \end{aligned}$$

and hence the graph satisfies Hall's condition. This completes the proof of the existence of the matching, and hence the induction proof is complete.

So, A can be written as a sum of permutation matrices.

2. Here, we will give an example of a graph which is color critical with chromatic number 3. Consider a pentagon, i.e a 5-cycle. It is easy to see that the chromatic number of a pentagon is 3. Deleting a vertex results in a path of length 3, and it is clear that the chromatic number of a path of length 3 is 2. So, a pentagon is a valid example.

3. Here, we will prove Hall's theorem from Tutte's theorem.

Let G be a bipartite graph with bipartition (X, Y).

First, suppose G has a matching M that saturates every vertex of X. Then, if $S \subset X$, then it is easy to see that $|N(S)| \geq |S|$ (because for every vertex in X we can find a corresponding vertex in $N(S)$ via the matching), and hence one direction of the theorem is trivial.

Conversely, suppose G satisfies Hall's condition, so that $|Y| \geq |X|$. Define H to be the following graph: if $|V(G)|$ is odd, then add a vertex in Y and complete Y, i.e add edges so that the Y-induced graph is complete (if $|V(G)|$ is even then we don't add a vertex to Y, we just complete it). We will now show that G has a matching saturating X if and only if the graph H has a perfect matching.

First, suppose H has a perfect matching M. This means that all vertices in X are saturated. Moreover, we never added a new edge to H between X and Y, and hence M is actually a matching of G, so that M is a matching of G that saturates X.

Conversely, suppose G has a matching that saturates X. Now, if $|X| = |Y|$, then it this matching is a perfect matching of H. If $|Y| > |X|$, then two cases are possible: If both $|X|$ and $|Y|$ are even (or both odd), then no new vertex was added in H, and hence the remaining vertices of Y that are not matched to any vertices in H can be paired up, and hence matched (because Y is a clique in H). If $|X| + |Y|$ is odd (i.e their parities are different), then a vertex was added in H, and hence again the remaining vertices of Y (plus the new vertex) that are not matched to any vertex of X can be paired up, and hence matched. So, in any case, a perfect matching of H has been found.

Finally, since G satisfies Hall's condition, we will show that H satisfies Tutte's condition, and hence this will prove Hall's theorem (by the fact we proved in the previous paragraph). For the sake of contradiction, suppose H does not satisfy Tutte's condition, and let T be some subset of $V(H)$ such that $o(H - T) > |T|$, where $o(\cdot)$ denotes the number of odd components. Since Y is a clique in H, there is some component C of $H - T$ which contains all vertices of $Y - T$, and the rest of the components are singleton sets in X. Let k be the number of singleton sets. If $|C|$ is even, then $o(H - T) = k$, but observe that $k \leq |T|$, which is a contradiction to the assumption that $o(H - T) > T$. So, $|C|$ must be odd.

Now, we have $|V(H)| = |T| + k + |C|$. Since $|C|$ is odd and $|V(H)|$ is even, we see that $|T| + k$ is odd. Also, $o(H - T) = k + 1 > |T|$ and hence $k \geq |T|$. Since $|T| + k$ is odd, it follows that $k > |T|$, but if we put S to be the set of these k vertices, then we have that $N(S) \subset T \cap Y$, and hence $|N(S)| \leq |T| < k = |S|$, which contradicts the fact that G satisfies Hall's condition. So, H must satisfy Tutte's theorem. This completes the proof of Hall's theorem from Tutte's theorem.

4. Let X be a largest independent set in G, and let $\alpha(G) = |X|$. We will show that V/X is a minimum vertex covering of G.

First, suppose uv is an edge in G. It follows that both u and v cannot be in X, and hence atleast one of them is in V/X, which proves that V/X is a vertex cover. To prove that it is a minimal vertex covering, let K be the set of vertices that lie on an edge such that one vertex of the edge lies in X and the other lies in V/X. We will prove that K contains atleast $|V/X|$ vertices, which will prove that V/X is a minimum covering. For the sake of contradiction, suppose K contains less than $|V/X|$ vertices, and hence there is some vertex v in V/X that is not in K. But, since X is a largest independent set, there must be some edge between v and and some vertex in X, contradicting the fact that v is not in K. This completes the proof.

Hence, it follows that $\alpha(G) + \tau(G) = n$.

5. Here, we will deduce Hall's theorem from Konig's theorem. So, suppose G is a bipartite graph with bipartition (X, Y).

One direction of Hall's theorem is trivial to prove (as we did in problem **3.**). So, we will only prove the harder direction.

Suppose G satisfies Hall's criterion. It then follows that $|Y| \geq |X|$ (because $N(X) \subset Y$ and $|N(X)| \geq |X|$). So it follows that X is a covering of G. Moreover, because $|N(v)| \geq 1$ for each $v \in X$, it follows that X is a minimum covering of G. Since G is bipartite, it follows that $|M| = |X|$, where M is the maximal matching in G.

So, it follows that G has a matching that saturates every vertex in X, proving Hall's theorem.

6. Suppose G is a bipartite graph, and let uv be an edge.

Suppose, for the sake of contradiction, that neither u nor v has the given property. So, we can find a maximum matchings M_1 and M_2 such that u is M_1-unsaturated and v is M_2 unsaturated. Moreover, because the matchings are maximum, v must be M_1-saturated and u must be M_2-saturated. Consider the symmetric difference $M_1 \Delta M_2$, and consider the graph induced by this symmetric difference.

In this graph, every vertex has degree atmost 2, because atmost one edge of M_1 can be incident to a vertex, and atmost one edge of M_2 can be incident to a vertex. So, the graph can be written as a union of disjoint even length cycles (cycle cannot be odd because the cycle will have alternate edges in M_1 and M_2, and odd cycle will have two adjacent edges of the same matching, which is not possible) and disjoint paths. Now, observe that the degrees of u and v are both 1, and hence u lies on a path. Let T be this path. Now, the length of T cannot be odd (i.e the first and last edges cannot both belong to M_2), because there are no M_1 augmented paths in G (Berge's theorem). So, it follows that T must be of even length. Also, the last vertex of this path cannot be v, because otherwise $T \cup \{uv\}$ will be an odd length cycle in G, which is not possible because G is bipartite.

Finally, consider the path $vu \cup T$ in G. Clearly, this is an M_2-alternating path (because $uv \ /M_2$), and hence this is an M_2- augmenting path (because the length of T was even), which contradicts Berge's theorem. This completes the proof via contradiction.

7. Suppose G is a bipartite graph with bipartition $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_n\}$. Consider the matrix M whose ij^{th} entry is one if and only if $a_i b_j$ is an edge. Also, suppose $\det(M) \neq 0$. We know that

$$\det(M) = \sum_{\sigma} \epsilon(\sigma) M_{1\sigma(1)} ... M_{n\sigma(n)}$$

where σ ranges over all permutations of $[n]$ and ϵ is the sign of the permutation. Since the determinant is non-zero, atleast one of these terms is non-zero, i.e atleast one of the products $M_{1\sigma(1)} ... M_{n\sigma(n)}$ is non-zero, which means a perfect matching of G exists (where the edges of the matching are $i\sigma(i)$).

10. Extremal Graph Theory

First, let us start with a discussion about bipartite graphs. We know that a graph G is bipartite if and only if it contains no odd cycle.

Now, consider the following simple bound:

Theorem 10.1. Let G be a graph on n vertices. If G is bipartite, then

$$|E| \leq \left\lfloor \frac{n^2}{4} \right\rfloor$$

Proof: This is a very simple bound. Suppose A and B are the bipartite components. So, $|A| + |B| = n$, and the maximum number of edges between these components is $|A||B|$. Clearly, we have that $|A|, |B| \leq \frac{n}{2}$, and hence $|A||B| \leq \frac{n^2}{4}$, proving the claim.

By the above theorem, if G has more than $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, then it must have an odd cycle. The following theorem, called *Mantel's theorem*, asserts a stronger implication:

Theorem 10.2. Mantel's theorem: If

$$|E| \geq \left\lfloor \frac{n^2}{4} \right\rfloor + 1$$

then G has a triangle.

Proof: We will prove this by induction on n.

If $n = 3$, then the theorem is clear. So, suppose the statement is true for some $n \in \mathbb{N}$, and we will prove it for $n+1$. Consider a graph on $n+1$ vertices having atleast $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor + 1$ vertices. Take any two vertices x and y, such that xy is an edge. If x and y share a common vertex, then we are done. Otherwise, they don't share a common vertex. Then, remove x and y from G to get a resultant graph G'. The number of edges $|E'|$ in G' will be

$$|E'| = |E| - d(x) - d(y) + 1$$

Observe that $d(x) + d(y) - 1 \leq n - 1$ and hence we have that

$$\begin{aligned}
|E'| &\geq |E| + 1 - n \\
&\geq \left\lfloor \frac{(n+1)^2}{4} \right\rfloor - \frac{4(n-1)}{4} + 1 \\
&= \left\lfloor \frac{(n+1)^2}{4} - \frac{4(n-1)}{4} \right\rfloor + 1 \\
&= \left\lfloor \frac{n^2 + 1 + 2n - 4n + 4}{4} \right\rfloor + 1 \\
&= \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 2
\end{aligned}$$

and by induction G' contains a triangle. Hence we are done.

We next ask the following question: describe all graphs on n vertices having exactly $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges and not having a triangle.

Clearly, one such graph is the complete bipartite graph. We claim that the complete bipartite graph is the only such possibility:

Theorem 10.3.

11. Exercises on page 259

(4). Let G be a bipartite graph in which every vertex of X has degree at least as large as the degree of any vertex in Y. We show that there is a matching M saturating every vertex in X. We will show that G satisfies Hall's condition.

Let $S \subset X$ be any non-empty subset of X. For the sake of contradiction, suppose $|N(S)| < |S|$. Suppose $|S| = x$, and let $|N(S)| = y$, $y < x$. Suppose $x = ky + r$, where $0 \leq r < y$. We consider two cases.

(1) Suppose $x = ky$, where $k \geq 2$. Because each vertex in S is connected to some vertex in $N(S)$, there is atleast one vertex in $N(S)$ of degree greater than or equal to $k \geq 2$. So, each vertex in S has degree atleast 2. Now, again, by

the pigeon hole principle, this means there is some vertex in $N(S)$ with degree atleast $2k$. Continuing this way, we can conclude that the degree of some vertex in $N(S)$ with arbitrarily large degree, a contradiction.

(2) Suppose $x = ky + r$, with $r > 0$, $k \geq 1$. We can now keep invoking the php as above.

(5). This can be done by induction and the formula

$$p(n, G) = p(n, G - uv) - p(n, G/uv)$$

where uv is an edge in G, and G/uv represents edge contraction.

(6). This can be easily done by induction. The base case with $n = 2$ is clear. For the inductive case, consider a triangle free graph with chromatic number n. Then, for any vertex $x \in G$, make a new vertex x' whose neighbors are exactly those of x. Do this for all vertices of G. Finally, add a new vertex y whose neighbors are all the newly added vertices. This completes the proof.