# TOPOLOGY

SIDDHANT CHAUDHARY

These are my course notes for the course TOPOLOGY that I undertook in my fourth semester.

## Contents



*Date*: January 2021.

# 1. Basic Concepts

1.1. **Introductory Definitions.** Let $X$ be any set. A *topology* $J$ on $X$ is a collection of subsets of $X$ satisfying the following.

(1) $\phi, X \in J$.
(2) $J$ is closed under arbitrary unions.
(3) $J$ is closed under finite intersections.

**Example 1.1.** Consider the set of real numbers $\mathbb{R}$, and let the topology on $\mathbb{R}$ be the collection of all open intervals in $\mathbb{R}$. It is then easily seen that these open sets satisfy the three axioms in the definition of a topology, and this is called the *standard topology* on $\mathbb{R}$.

**Example 1.2.** Let $X$ be any set. The power set $\mathscr{P}(X)$ forms a topology on $X$, and this is called the *discrete topology*. Analogously, there is something called the *indiscrete topology*, which is just the collection $\{\phi, X\}$. These are respectively the largest and the smallest topologies that we can have on a set $X$.

**Example 1.3.** Let $X$ be any set, and put

$$J := \{U \subseteq X \mid X \setminus U \text{ is finite}\} \cup \{\phi\}$$

Using De-Morgan's Laws, it is easy to see that $J$ is a topology on $X$, and this is called the *Zariski Topology*. In the above definition, we can replace the word *finite* by the word *countable* as well. That topology is called the *countable complement topology*.

**Exercise 1.1.** Compare the Zariski topology and the standard topology on $\mathbb{R}$.

**Solution.** We show that the Zariski Topology is *coarser* (i.e smaller) than the standard topology on $\mathbb{R}$. To show this, suppose $U$ is an open subset in the Zariski topology, so that $\mathbb{R} \setminus U$ is a finite set. Suppose $\mathbb{R}\setminus = \{a_1, ..., a_n\}$, where we assume without loss of generality that $a_1 < a_2 < ... < a_n$. So, it follows that

$$U = (-\infty, a_1) \cup (a_1, a_2) \cup ... \cup (a_n, \infty)$$

and hence $U$ is a member of the standard topology. So, it follows that the Zariski topology is coarser. It is easy to see that these topologies are *not* equivalent.

**Example 1.4.** Let $(X, d)$ be any metric space. For $x_0 \in X$ and $r \in \mathbb{R}$ with $r > 0$, we define

$$B_{x_0}(r) := \{x \in X \mid d(x, x_0) < r\}$$

These are called *open balls*. Then the set of all unions of open balls in $X$ form a topology on $X$, i.e the set

$$J_{\text{metric}} := \{\text{union of open balls in } X\}$$

is a topology on $X$. Let us now prove this. It is clear that $\phi, X \in J_{\text{metric}}$. By the definition of $J_{\text{metric}}$, it is closed under taking arbitrary unions. Finally, we need to show the closure under finite intersections. Suppose $M$ is a finite intersection of unions of open balls, i.e

$$M = \left(\bigcup_{\alpha_1 \in J_1} B_{\alpha_1}\right) \cap \left(\bigcup_{\alpha_2 \in J_2} B_{\alpha_2}\right) \cap ... \cap \left(\bigcup_{\alpha_n \in J_n} B_{\alpha_n}\right)$$

where $J_1, ..., J_n$ are indexing sets and each $B_{\alpha_i}$ is an open ball in $X$. It is then easy to see that

$$M = \bigcup_{(\alpha_1,...,\alpha_n)\in J_1\times...\times J_n} (B_{\alpha_1} \cap ... \cap B_{\alpha_n})$$

Now here is the key property to use: any *finite* intersection of open balls in $X$ is itself a union of open balls, and this is immediately seen because if $x \in B_{\alpha_1} \cap ... \cap B_{\alpha_n}$, then there is a ball $B$ such that $x \in B \subseteq B_{\alpha_1} \cap ... \cap B_{\alpha_n}$. Then, we can simply take the union of all such open balls $B$ ranging over all $x \in B_{\alpha_1} \cap ... \cap B_{\alpha_n}$, and that expresses this finite intersection as a union of open balls. So, it then follows that $M$ is a union of open balls, and this proves that $J_{\text{metric}}$ is indeed closed under finite intersections. So, $J_{\text{metric}}$ forms a topology.

**Example 1.5.** Let $X$ be any set. Then, the discrete topology on $X$ is a metric topology, where the metric is simply the discrete metric on $X$. On the other hand, the indiscrete topology on $X$ is a metric topology if and only if $|X| = 1$. This is easy to see.

**Definition 1.1.** Let $X$ be a topological space and let $Y \subseteq X$ be a subset. Define

$$J_Y := \{U \cap Y \mid U \text{ is open in } X\}$$

Then $J_Y$ is a topology on $Y$ and is called the *subspace topology* on $Y$ (the fact that $J_Y$ is indeed a topology on $Y$ is relatively straightforward to check).

**Definition 1.2.** Let $X$ be a set. A collection $\mathcal{B}$ of subsets of $X$ is called a *basis for a topology* on X if:

(1) $X$ is the union of elements of $\mathcal{B}$.
(2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

**Definition 1.3.** Let $\mathcal{B}$ be a basis for a topology on $X$. Then, consider the set $J_{\mathcal{B}}$ defined as below.

$$J_{\mathcal{B}} := \{U \subseteq X \mid \forall x \in U\, \exists B \in \mathcal{B} \text{ s.t } x \in B \subseteq U\}$$

$J_{\mathcal{B}}$ is said to be the *topology generated* by $\mathcal{B}$.

**Proposition 1.1.** *Let $\mathcal{B}$ be a basis for a topology on $X$, and let $J_{\mathcal{B}}$ be the topology generated by $\mathcal{B}$. Then $J_{\mathcal{B}}$ is indeed a topology on $X$.*

*Proof.* It is vacuously true that $\phi \in J_{\mathcal{B}}$, and since $\mathcal{B}$ is a basis for a topology on $X$, it follows that $X \in J_{\mathcal{B}}$. Next, we show that $J_{\mathcal{B}}$ is closed under taking arbitrary unions. But we can show something stronger: any element of $J_{\mathcal{B}}$ is a union of sets in $\mathcal{B}$. To show this, suppose $U \in J_{\mathcal{B}}$, and let $x \in U$. Then, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Taking the union of all such $B$ as $x$ ranges over $U$, we see that $U$ is the union of sets in $\mathcal{B}$. So, it follows that $J_{\mathcal{B}}$ is closed under taking arbitrary unions. Finally, we show that $J_{\mathcal{B}}$ is closed under taking intersections. So, let $U_1, ..., U_n \in J_{\mathcal{B}}$, so each $U_i$ is a union of elements of $\mathcal{B}$. Then, proceed as in **Example** 1.4 to show that $U_1 \cap ... \cap U_n$ is a union of sets in $\mathcal{B}$, and this is where property (2) of **Definition** 1.2 comes into play. This completes the proof. ■

**Proposition 1.2.** *Let $X$ be a set and let $\mathcal{B}$ be a basis for a topology on $X$. Then, a set $U$ is open in the topology $J_{\mathcal{B}}$ if and only if $U$ is equal to the union of sets in $\mathcal{B}$.*

*Proof.* The forward direction of the claim was proven in **Proposition** 1.1 above. The backward direction is immediate from the fact that $J_{\mathcal{B}}$ is a topology on $X$. ■

**Example 1.6.** Let $\mathcal{B}' = \{[a,b) \mid a,b \in \mathbb{R}\}$. This is a basis for some topology on $\mathbb{R}$, and this is called the *lower limit topology* $\mathbb{R}_l$. We will show that the lower limit topology is strictly finer than the standard topology on $\mathbb{R}$. To show this, let $(a,b)$ be any interval in $\mathbb{R}$. We can write this interval as

$$(a,b) = \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right)$$

It is *strictly closed* because $[a,b)$ cannot be written as a union of open intervals in $\mathbb{R}$, because $[a,b)$ is *not* an open set with respect to the metric on $\mathbb{R}$ (and we know that the metric topology and the standard topology in $\mathbb{R}$ are equivalent).

**Example 1.7.** Here we introduce a topology on $\mathbb{R}$ which will be denoted by $\mathbb{R}_K$. Let

$$\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R}\} \cup \{(a,b) \setminus K \mid a,b \in \mathbb{R}\}$$

where

$$K = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$$

First, we show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. Clearly, $\mathbb{R}$ can be written as a union of sets in $\mathcal{B}$. Next, we show that property (2) of **Definition** 1.2 is satisfied. So, let $B_1, B_2 \in \mathcal{B}$. We have the following cases.

(1) $B_1 = (a,b)$ and $B_2 = (c,d)$ for some $a,b,c,d \in \mathbb{R}$. If $x \in B_1 \cap B_2$, then there is some open interval containing $x$ contained in $B_1 \cap B_2$. So this case can be handled.
(2) $B_1 = (a,b)$ and $B_2 = (c,d) \setminus K$. In this case, take $x \in B_1 \cap B_2$, and let $B$ be an open interval such that $x \in B \subseteq B_1 \cap (c,d)$. Then, just consider the interval the set $B \setminus K$. So this case is also handled.
(3) In this case, $B_1 = (a,b) \setminus K$ and $B_2 = (c,d) \setminus K$. This case is handled similar to case (2).

So, $\mathcal{B}$ is indeed a basis to a topology in $\mathbb{R}$. We will now show that $\mathbb{R}_K$ is *strictly finer* than the standard topology of $\mathbb{R}$. To show this, it is enough to show that $(-1,1) \setminus K$ *cannot* be written as a union of open intervals in $\mathbb{R}$. This is because $(-1,1) \setminus K$ is *not* open in the metric topology of $\mathbb{R}$ (look at the point 0).

1.2. **Product Topology.** Here we will introduce a way to construct topology on the cartesian product of sets.

**Proposition 1.3.** *Let $X, Y$ be topological spaces, and consider the set $X \times Y$. Let*

$$\mathcal{B} = \{U \times V \mid U \subseteq X, V \subseteq Y \text{ are open sets}\}$$

*Then, $\mathcal{B}$ is not a topology because it is not closed under union. However, $\mathcal{B}$ is a basis for a topology on $X \times Y$.*

*Proof.* We will only show that property (2) in **Definition** 1.2 holds, because property (1) clearly holds. But property (2) holds because if $U_1 \times V_1$ and $U_2 \times V_2$ are two sets in $\mathcal{B}$, then

$$U_1 \times V_1 \cap U_2 \times V_2 = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$$

because of the finite intersection closure property of topological spaces. This completes the proof. ■

**Definition 1.4.** The topology generated by the set $\mathcal{B}$ as in **Proposition** 1.3 is called the *product topology* on $X \times Y$. This extends to a finite product $X_1 \times \ldots \times X_n$ of topological spaces, where a basis for the topology on $X_1 \times \ldots \times X_n$ is given by

$$\mathcal{B} = \{U_1 \times \ldots \times U_n \mid U_i \subseteq X_i \text{ is open}\}$$

**Example 1.8.** We show that the product topology on $\mathbb{R}^2$ is the same as the metric topology on $\mathbb{R}^2$ given by the Euclidean metric. Note that the basis for the product topology is all sets of the form $A \times B$, with $A, B \subseteq \mathbb{R}$ open sets, and the basis for the metric topology is the set of all open balls in $\mathbb{R}^2$. Keeping this in mind, we do the following.

Let $B$ be any open ball in $\mathbb{R}^2$ centered at a point $x_0 \in \mathbb{R}^2$ containing a point $x \in \mathbb{R}^2$, and let the radius of $B$ be $r > 0$. So, consider the ball $B(x, r - d(x_0, x))$; observe that $0 < r - d(x_0, x) < r$, so this ball makes sense. Clearly, we have that

$$B(x, r - d(x_0, x)) \subseteq B$$

Put $\delta = r - d(x_0, x)$, so our ball of consideration is $B(x, \delta) \subseteq B$. Write $x = (x_1, x_2)$ where $x_1, x_2 \in \mathbb{R}$. Now, consider the product

$$\left(x_1 - \frac{\delta}{2}, x_1 + \frac{\delta}{2}\right) \times \left(x_2 - \frac{\delta}{2}, x_2 + \frac{\delta}{2}\right) =: A \times B$$

It is very easy to see that $A \times B \subseteq B(x, \delta) \subseteq B$. What we have just shown is that the product topology is *finer* than the metric topology.

Now, let us prove the converse, i.e the metric topology is *finer* than the product topology, and that will prove the equivalence. So, let $A \times B$ be any product with $A, B$ being open subsets of $\mathbb{R}$, and let $x \in \mathbb{R}^2$ be a point in $A \times B$. Write $x = (x_1, x_2)$, so that $x_1 \in A$ and $x_2 \in B$. Since $A$ and $B$ are open in $\mathbb{R}$, there are $\delta_1, \delta_2 > 0$ such that

$$(x_1 - \delta_1, x_1 + \delta_1) \subseteq A \quad , \quad (x_2 - \delta_2, x_2 + \delta_2) \subseteq B$$

Put $\delta = \max\{\delta_1, \delta_2\}$. Then, it is straightforward to check that

$$x \in B(x, \delta) \subseteq A \times B$$

So, it follows that the metric topology is *finer* than the metric topology. This completes our proof.

**Exercise 1.2.** Let $X, Y$ be two topological spaces. Let $\mathcal{B}_X$, $\mathcal{B}_Y$ be bases for the topologies on $X$ and $Y$ respectively. Let

$$\mathcal{B}' := \{B_1 \times B_2 \mid B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_Y\} \subseteq \mathcal{B}$$

Show that $\mathcal{B}'$ generates the product topology on $X \times Y$.

**Solution.** Let's first show that $\mathcal{B}'$ is a basis for some topology on $X \times Y$, and then we will show that the topology generated is the product topology. First, we show that $X \times Y$ can be written as a union of elements of $\mathcal{B}'$. So, suppose

$$\bigcup_{\alpha \in I} B_\alpha = X \quad , \quad \bigcup_{\beta \in J} B_\beta = Y$$

where $B_\alpha \in \mathcal{B}_X$ for each $\alpha$, and $B_\beta \in \mathcal{B}_Y$ for each $\beta$. So, it follows that

$$\bigcup_{(\alpha,\beta) \in I \times J} B_\alpha \times B_\beta = X \times Y$$

and this shows that $X \times Y$ is a union of elements of $\mathcal{B}'$. Next, suppose $B_1 \times B_2 \in \mathcal{B}'$ and $B_1' \times B_2' \in \mathcal{B}'$, and suppose $(x, y) \in B_1 \times B_2 \cap B_1' \times B_2'$. Then, there is some

$B_1'' \in \mathcal{B}_X$ such that $x \in B_1'' \subseteq B_1 \cap B_1'$. Similarly, there is some $B_2'' \in \mathcal{B}_Y$ such that $y \in B_2'' \subseteq B_2 \cap B_2'$. So, it follows that

$$(x, y) \in B_1'' \times B_2'' \subseteq B_1 \times B_2 \cap B_1' \times B_2'$$

and hence it follows that $\mathcal{B}'$ is indeed a basis for some topology on $X \times Y$. Now, we show that this is nothing but the product topology. Note that it is clear that $B'$ is coarser than the product topology. Now, suppose $U_1 \times U_2$ is any basic open set in the product topology. So, we can write

$$\bigcup_{\alpha \in I} B_\alpha = U_1 \quad , \quad \bigcup_{\beta \in J} B_\beta = U_2$$

and hence

$$\bigcup_{(\alpha,\beta) \in I \times J} B_\alpha \times B_\beta = U_1 \times U_2$$

So, it follows that the product topology is coarser than $\mathcal{B}'$. This completes our proof.

**Definition 1.5.** Let $X, Y$ be topological spaces. Then a map $f : X \to Y$ is said to be *continuous* if $f^{-1}(U)$ is open in $X$ for all open sets $U \subseteq Y$.

**Exercise 1.3.** Let $X, Y$ be topological spaces. Then show that the product topology is the coarsest topology on $X \times Y$ such that both projections $p_1 : X \times Y \to X, p_2 : X \times Y \to Y$ are continuous.

**Solution.** Let $\mathcal{B}$ be any topology on $X \times Y$ such that both the projection maps $p_1, p_2$ are continuous. It is enough to show that the set $U_1 \times U_2$ is open in $X \times Y$, where $U_1, U_2$ are open sets in $X$ and $Y$ respectively. But, observe that

$$U_1 \times Y = p_1^{-1}(U_1) \quad , \quad X \times U_2 = p_2^{-1}(U_2)$$

So, both sets $U_1 \times Y$ and $X \times U_2$ are open in $X \times Y$. So, it follows that

$$U_1 \times U_2 = U_1 \times Y \cap X \times U_2$$

is open in $X \times Y$, and this completes the proof.

1.3. **Order Topology.** Let $X$ be any *ordered set*. An example will be $X = \mathbb{R}$. For $a, b \in X$, we have the usual notion of intervals.

$$(a, b) := \{x \in X \mid a < x < b\}$$
$$[a, b) := \{x \in X \mid a \leq x < b\}$$

Now, let $\mathcal{B} = \{(a, b) \mid a, b \in X\}$. The question is whether $\mathcal{B}$ is a basis for some topology on $X$. The answer is *no* in general; as a counter example take $X = \mathbb{Z}_{>0}$. Because $X$ has a least element, namely the integer 1, we cannot write $X$ as a union of sets in $\mathcal{B}$. To remedy this situation, we add new sets to $\mathcal{B}$ as follows.

$$\mathcal{B} = \{(a, b) \mid a, b \in X\} \cup \{[a_0, b) \mid b \in X\} \cup \{(a, b_0] \mid a \in X\}$$

where $a_0$ is the least element of $X$ (if it exists), and similarly $b_0$ is the greatest element of $x$ (if it exists). Then, it turns out that $\mathcal{B}$ is a basis for a topology on $X$, and this is relatively easy to verify. The topology generated by the set $\mathcal{B}$ is called the *order topology* on $X$.

**Example 1.9.** Consider the order topology on $\mathbb{R}$; it turns out that this is the same as the standard topology on $\mathbb{R}$. This is clear, because $\mathbb{R}$ does not have any least element. So, there are no sets of the form $[a_0, b)$ or $(a, b_0]$ in the basis.

**Example 1.10.** Let $X = \mathbb{R}^2$, and we know that the metric topology on $X$ is the same as the product topology. We impose the so called *dictionary order* on $X$. For $(a,b),(c,d) \in \mathbb{R}^2$, we say $(a,b) \leq (c,d)$ if $a < c$ or $a = c$ and $b \leq d$. In other words, this is just the lexicographic ordering. The topology induced by this ordering of $\mathbb{R}^2$ is called the *dictionary order topology.* It can be checked that this topology is *finer* than the standard topology, but the two are not equivalent.

**Example 1.11.** Let $X = \mathbb{Z}_{>0}$ with the usual ordering. We show that the order topology on $X$ is the discrete topology. To show this, it is enough to show that every singleton set is open. If $d \in \mathbb{Z}_{>0}$ such that $d > 1$, then note that $(d-1, d+1) = \{d\}$. Moreover, $\{1\} = [1,2)$. Hence, it follows that this is the discrete topology.

**Example 1.12.** Consider the set $X = \{1,2\} \times \mathbb{Z}_{>0}$ with the dictionary ordering. In the case, the order topology is *not* the discrete topology. To see this, observe that the set $\{(2,1)\}$ is not open in this topology.

**Example 1.13.** Let $I = [0,1] \subseteq \mathbb{R}$ and let $X = I \times I \subseteq \mathbb{R}^2$. Then consider the dictionary order topology on $X$ and the subspace topology on $X$ coming from the dictionary order topology on $\mathbb{R}^2$. We claim that these two are *not* equivalent. To see this, note that the vertical line segment between $(0,0)$ and $(0,1)$ in $X$ is open in the subspace topology, but it is *not* open in the dictionary order topology. It is also not hard to see that the subspace topology is infact *finer* than the dictionary order topology.

1.4. **The Closure and Interior.** Let $X$ be a topological space, and let $A \subseteq X$ be a subset. We define the *interior* of $A$ as

$$A^\circ := \text{union of all open sets contained in } A$$

Similarly, the *closure* of $A$ is defined as

$$\overline{A} := \text{intersection of all closed sets containing } A$$

Clearly, these definitions imply that

$$A^\circ \subseteq A \subseteq \overline{A}$$

**Definition 1.6.** Let $X$ be any topological space and let $A \subseteq X$ be a subset. A point $x \in X$ is a *limit point* of $A$ if every open set $U \subseteq X$ containing $x$ intersects $A$ in some point other than $x$, i.e if $U \subseteq X$ is open and $x \in U$, then $U \cap A \neq \phi, \{x\}$.

**Proposition 1.4.** *Let $A \subseteq X$ be a subset of a topological space $X$. Let $A'$ denote the set of limit points of $A$. Then $\overline{A} = A \cup A'$.*

*Proof.* First, let $x \in A \cup A'$. If $x \in A$, then clearly $x \in \overline{A}$. So suppose $x \in A'$, i.e $x$ is a limit point of $A$. We want to show that $x \in \overline{A}$. So, let $C$ be any closed subset of $X$ containing $A$, and we want to show that $x \in C$. For the sake of contradiction, suppose $x \in C^c$. Because $C^c$ is open, it follows that $C^c \cap A$ contains a point different from $x$, i.e $C^c \cap A \neq \phi$. But this contradicts the fact that $A \subseteq C$, and hence $x \notin C^c$, so that $x \in C$. This shows that $x \in \overline{A}$, and hence $A \cup A' \subseteq \overline{A}$.

To prove the reverse inclusion, suppose $x \in \overline{A}$, i.e $x$ is contained in every closed set $C$ containing $A$. In addition, suppose $x \notin A$. Then, we show that $x \in A'$. So, let $U$ be any open subset of $X$ containing $x$. We need to show that $U \cap A \neq \phi, \{x\}$. Because by assumption $x \notin A$, it is enough to show that $U \cap A \neq \phi$. For the sake of contradiction, suppose $U \cap A = \phi$, and this implies that $A \subseteq U^c$. Since $U^c$ is closed, this implies that $x \in U^c$, and this is clearly a contradiction. So, it follows that $U \cap A \neq \phi$, i.e $x \in A'$. We have just shown that $\overline{A} \subseteq A \cup A'$. This completes the proof. ■

**Exercise 1.4.** Show that $x$ is a limit point of $A$ if and only if $x \in \overline{A - \{x\}}$.

**Solution.** First, suppose $x$ is a limit point of $A$, and let $C$ be any closed set containing $A - \{x\}$. We need to show that $x \in C$. For the sake of contradiction, suppose $x \notin C$, so that $x \in C^c$, which is an open set. So, it follows that $C^c \cap A \neq \phi, \{x\}$, and this contradicts the fact that $A - \{x\} \subseteq C$. To, $x \in C$ and hence $x \in \overline{A - \{x\}}$.

Conversely, suppose $x \in \overline{A - \{x\}}$. We show that $x$ is a limit point of $A$. So let $U$ be any open subset of $X$ containing $x$. We need to show that $U \cap A \neq \phi, \{x\}$, i.e $U$ contains a point other than $x$. Because $x \in \overline{A - \{x\}}$, we see that either $x \in A - \{x\}$ or $x \in (A - \{x\})'$. Clearly, we see that $x \in (A - \{x\})'$. This implies that $U \cap (A - \{x\}) \neq \phi$, i.e $U \cap \{A - \{x\}\}$ contains a point other than $x$, and hence $U \cap A$ contains a point other than $x$. This shows that $x \in A'$, and completes our proof.

**Definition 1.7.** Let $X$ be a topological space. Then $X$ is said to be *Hausdorff* if for any $x, y \in X$, there are open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$. An open set containing a point $x$ is called a *neighborhood* of $x$.

**Example 1.14.** $\mathbb{R}$ with the standard topology is clearly Hausdorff. More generally, $\mathbb{R}^n$ with the product topology is Hausdorff. Also, the lower limit topology $\mathbb{R}_l$ and the topology $\mathbb{R}_K$ that we saw before are also Hausdorff.

**Example 1.15.** Any discrete topology is clearly Hausdorff. The indiscrete topology is *not* Hausdorff if the space has more than two points.

**Example 1.16.** Let $(X, d)$ be any metric space and let the topology in question be the metric topology. We show that $X$ is always a Hausdorff space. To show this, if $x, y \in X$ are any two points, then we can take balls centered at $x$ and $y$ with radius $d(x, y)/2$.

**Example 1.17.** Let $X$ be any ordered set, and we show that any order topology is always Hausdorff. So, let $x, y \in X$ and without loss of generality we assume that $x < y$. Then, we a couple of cases, and symmetric cases can be handled similarly.

(1) In the first case, $x$ is the smallest element of $X$ and $y$ is the greatest element of $X$. This case has two subcases: in the first subcase, there is an element $z$ between $x, y$, i.e $x < z < y$. So, the required open sets are $[x, z)$ and $(z, y]$. In the second subcase, there is no element between $x$ and $y$. Here, the required open sets are simply $[x, y)$ and $(x, y]$.

(2) In the second case, $x$ is the smallest element of $X$ and $y$ is not the greatest element of $X$, i.e there is some $y'$ such that $y < y'$. Again, there are two subcases. In the first subcase, there is an element $z$ between $x$ and $y$, i.e $x < z < y$. In this case, the required open sets are $[x, z)$ and $(z, y')$. In the second subcase, there is no element between $x$ and $y$. In this case, the required open sets are $[x, y)$ and $(x, y')$.

(3) In the third case, $x$ is not the least element of $X$ and $y$ is not the greatest element of $X$. This case can be handled in a similar way as above.

So, it follows that the order topology is indeed Hausdorff.

**Example 1.18.** Let $X$ be any infinite set. Then $X$ is *not* Hausdorff in the finite complement topology. Suppose, for the sake of contradiction, that $X$ is Hausdorff, and let $x, y \in X$. So, there are disjoint open sets $U, V$ in $X$ such that $x \in U$ and $y \in V$. Now because $U$ is open, $U^c$ is finite, and since $V \subseteq U^c$, it follows that $V$ is finite. But, $V$ is also open, which means that $V^c$ is finite. However, because $X$ is an infinite set, both $V$ and $V^c$ cannot be finite. Hence, $X$ is not a Hausdorff space.

**Proposition 1.5.** *Let $X$ be any Hausdorff space. Then any finite subset of $X$ is closed.*

*Proof.* Note that it suffices to show that every singleton set is closed. Let $x \in X$. To show $\{x\}$ is closed, it is equivalent to show $X - \{x\}$ is open. For $Y \in X - \{x\}$, choose a neighborhood $U_y$ of $y$ such that $x \notin U_y$. Then $X - \{x\} = \bigcup_{y \neq x} U_y$ is open, and this completes the proof. ■

The conclusion of the proposition is true with weaker hypothesis than Hausdorfness. And that is what we will define now.

**Definition 1.8.** A topological space $X$ is said to satisfy the $T_1$ *property* if given $x \neq y \in X$, there are neighborhoods $U_x$ and $U_y$ of $x$ and $y$ respectively such that $y \notin U_x$ and $x \notin U_y$.

**Remark 1.5.1.** Conditions like the $T_1$ property are called *separation axioms*. The name comes from the fact that these properties measure the extent upto which points can be separated using open sets in a given space. The Hausdorff property is called the $T_2$ *property*.

**Example 1.19.** If $X$ is a Hausdorff space, then clearly it has the $T_1$ property. However, the converse of this is *not* true in general. As a counter example, let $X$ be an infinite set, and consider the finite complement topology on $X$. From **Example** 1.18, we know that $X$ is not a Hausdorff space. But it is not hard to see that $X$ has the $T_1$ property.

**Exercise 1.5.** Show that $X$ is $T_1$ if and only if every finite set in $X$ is closed.

**Solution.** First, suppose that $X$ is $T_1$. It is enough to show that every singleton set $\{x\}$ is closed, i.e $X - \{x\}$ is open. To do this, let $y \in X - \{x\}$, and let $U_y$ be an open subset of $X$ that does not contain $x$. So, we see that

$$X - \{x\} = \bigcup_{y \in X - \{x\}} U_y$$

and hence $X - \{x\}$ is open, so that $\{x\}$ is closed. Conversely, suppose every finite subset of $X$ is closed, and let $x, y \in X$ be any two distinct points. Clearly, $\{x\}$ and $\{y\}$ are both closed sets, and hence their complements are open. The complements can taken to be the required neighborhoods.

**Example 1.20.** Let $R$ be a non-zero commutative ring with unity. Let

$$X = \mathrm{Spec}(R) := \{\text{prime ideals of } R\}$$

We define the *Zariski topology* on $X$ as follows: if $I \subseteq R$ is an ideal, put

$$V(I) := \{P \in X \mid I \subseteq P\}$$

and we define $V(I)$ to be a closed set. Let us show that this is indeed a topology on $X$. Because $R$ is an ideal and is not contained in any prime ideal, we see that $\phi$ is a closed set in this topology. Similarly, since the zero ideal is contained in every prime ideal, we see that $X$ is a closed set in this topology. Since, we show that a finite union of closed sets is closed. So, let $I_1, ..., I_n$ be finitely many ideals of $R$, and consider the corresponding sets $V(I_1), ..., V(I_n)$. We claim that

$$V(I_1) \cup V(I_2) \cup ... \cup V(I_n) = V(I_1 \cap ... \cap I_n)$$

To show this, suppose $I_i \subseteq P$ for some $1 \leq i \leq n$ and $P \in X$. Then clearly, $I_1 \cap I_2 \cap ... \cap I_n \subseteq P$. Conversely, suppose $I_1 \cdot I_2 \cdot ... \cdot I_n \subseteq P$ for some prime ideal $P$.

For the sake of contradiction, suppose $P$ does not contain $I_i$ for *any* $1 \leq i \leq n$. Then, there are elements $x_1, ..., x_n$ such that $x_i \in I_i$ and $x_i \notin P$ for every $1 \leq i \leq n$. But clearly, we see that $x_1...x_n \in I_1 \cap ... \cap I_n$, and hence $x_1...x_n \in P$. Because $P$ is prime, this implies that $x_i \in P$ for some $I$, a contradiction. So, it follows that $I_i \subseteq P$ for some $1 \leq i \leq n$, and this proves our claim. So, it follows that finite unions of closed sets are closed.

Finally, we show that an arbitrary intersection of closed sets is closed. So, let

$$\bigcap_\alpha V(I_\alpha)$$

be an arbitrary intersection of closed sets. We claim that

$$\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right)$$

So suppose there is some prime ideal $P$ such that $I_\alpha \subseteq P$ for all $\alpha$. It is then clear that $\sum_\alpha I_\alpha \subseteq P$. The converse to this is trivial. Hence, this proves that this is a valid topology on $X$.

**Exercise 1.6.** Find a ring $R$ such that $\text{Spec}(R)$ is not $T_1$ in the Zariski topology.

**Solution.** Let $R = \mathbb{C}[t]$, and we will show that $\text{Spec}(R)$ is not $T_1$ by showing that the set $\{0\}$ is *not* closed in $\text{Spec}(R)$, where 0 is the zero ideal (which we know is prime), and clearly this will contradict the property in **Exercise** 1.5. To show this, suppose $\{0\}$ is closed, i.e $\{0\} = V(I)$ for some ideal $I \subseteq \mathbb{C}[t]$. This implies that $I \subseteq 0$, i.e $I = 0$. But, we know that $V(0) = \text{Spec}(R) \neq \{0\}$, which is a contradiction. So, it follows that $\{0\}$ is not a closed set, and hence $\text{Spec}(R)$ is *not* $T_1$.

**Example 1.21.** Here is another example from ring theory: let $R = \mathbb{C}[t]$ Consider the statement of *Hilbert's Nullstellensatz*: *an ideal $I \subseteq \mathbb{C}[t]$ is maximal if and only if $I = (t - \alpha)$ for some $\alpha \in \mathbb{C}$*. From this, we can say that maximal ideals of $R$ are in bijective correspondence with $\mathbb{C}$. Now we have seen above that $\text{Spec}(R)$ is not $T_1$ in the Zariski Topology.

Now, let

$$X = \{\text{maximal ideals of } R\} \subseteq \text{Spec}(R)$$

Then, we show that the subspace topology on $X$ from the Zariski Topology on $\text{Spec}(R)$ is the same as the finite complement topology on $X$. To be completed.

**Proposition 1.6.** *Let $X$, $Y$ be any topological spaces.*

(1) *If $X$ is Hausdorff and $Z \subseteq X$, then $Z$ is also Hausdorff in the subspace topology.*
(2) *If $X, Y$ are Hausdorff, then $X \times Y$ is also Hausdorff.*

*Proof.* (1) is trivial. For (2), let $(x_1, y_1)$ and $(x_2, y_2)$ be any distinct points of $X \times Y$. Let $U_1, U_2$ be disjoint open neighborhoods of $x_1, x_2$ in $X$ and let $V_1, V_2$ be disjoint open neighborhoods of $y_1, y_2$ in $Y$. So, it follows that $U_1 \times V_1$ and $U_2 \times V_2$ are disjoint open neighborhoods of $(x_1, y_1)$, $(x_2, y_2)$ in $X \times Y$. So the claim is true. ■

## 1.5. Continuous functions.

We have already seen the definition of continuous functions in **Definition** 1.5. Let us look at some examples.

**Example 1.22.** Consider the identity map $f : \mathbb{R} \to \mathbb{R}_l$, where as before $\mathbb{R}_l$ is the lower limit topology. Clearly, this function is *not* continuous, because the inverse image of $[a, b)$ is itself, and this is *not* open in the standard topology. However, if we switch the domain and the codomain, i.e we consider the identity map $g : \mathbb{R}_l \to \mathbb{R}$, then this function is continuous, because $\mathbb{R}_l$ is finer than the standard topology.

**Definition 1.9.** A function $f : X \to Y$ is called a *homeomorphism* if $f$ is continuous, bijective and the inverse $f^{-1} : Y \to X$ is also continuous. In this case, $X, Y$ are said to be homeomorphic and we write $X \cong Y$.

**Remark 1.6.1.** A bijective continuous function is not necessarily a homeomorphism. Infact, **Example** 1.22 is a valid counterexample. Compare this situation with the one we usually see in algebra, for instance in vector space, ring or group homomorphisms. Another example is given below.

**Example 1.23.** Consider the unit circle $S^1$ equipped with the subspace topology induced by the metric topology on $\mathbb{R}^2$. Let $[0, 1)$ be equipped with the subspace topology induced by the standard topology on $\mathbb{R}$. Consider the function $f : [0, 1) \to S^1$ given by

$$f(t) = (\cos(2\pi t), \sin(2\pi t))$$

It is clear that $f$ is a continuous bijection. However, we show that $f$ is *not* a homeomorphisms. To see this, consider the open set $\left[0, \frac{1}{4}\right)$ in $[0, 1)$. Observe that $f(U)$ is the quarter of the boundary of the circle lying in the first quadrant, including the point $(0, 0)$, and this is clearly not open in $S^1$, i.e $f^{-1}$ is not a continuous function.

1.6. **Arbitrary Products.** Let $J$ be any indexing set, and let $\{X_\alpha\}_{\alpha\in J}$ be a collection of topological spaces. As a set, define

$$X := \prod_{\alpha\in J} X_\alpha = \{(x_\alpha)_{\alpha\in J} \mid x_\alpha \in X_\alpha\}$$

We can now define two topologies on $X$.

(1) The first is called the *product topology* on $X$. Here, basic open sets are of the form
$$\prod_{\alpha\in J} U_\alpha$$
where $U_\alpha \subseteq X_\alpha$ is open for all $\alpha \in J$, and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in J$.

(2) The second one is called the *box topology* on $X$. Here, everything is the same as above, except that we don't have the finiteness condition.

Clearly, if $J$ is a finite set, then the product and box topologies are the same.

**Exercise 1.7.** Let $\{X_\alpha\}$ be any family of Hausdorff spaces. Then

$$\prod_\alpha X_\alpha$$

is Hausdorff in both product and box topologies.

**Solution.** Let $(x_\alpha)$ and $(y_\alpha)$ be any two distinct points in the cartesian product. So, there is some $\beta$ such that $x_\beta \neq y_\beta$. So, there are disjoint open sets $U_{x,\beta}, U_{y,\beta}$ of $X_\beta$ containing $x_\beta$ and $y_\beta$ respectively. So, the two sets

$$\prod_\alpha J_{1,\alpha} \text{ and } \prod_\alpha J_{2,\alpha}$$

where $J_{1,\alpha} = X_\alpha$ for all $\alpha \neq \beta$ and $J_{1,\beta} = U_{x,\beta}$ and $J_{2,\alpha} = X_\alpha$ for all $\alpha \neq \beta$ and $J_{2,\beta} = U_{y,\beta}$ are disjoint open subsets of the box/product topologies that contain $(x_\alpha)$ and $(y_\beta)$ respectively.

**Proposition 1.7.** *Let $A$, $X_\alpha$, $\alpha \in J$ be topological spaces and let $f : A \to \prod_{\alpha \in J} X_\alpha$ be a map given by*

$$f(a) = (f_\alpha(a))_{\alpha \in J}, \quad f_\alpha : A \to X_\alpha$$

*and where $\prod_\alpha X_\alpha$ is given the product topology. Then, $f$ is continuous if and only if each $f_\alpha$ is continuous.*

**Remark 1.7.1.** This statement is only true if $\prod_\alpha X_\alpha$ is given the product topology, and it doesn't hold in the box topology. We will see why in the proof.

*Proof.* First, suppose $f$ is a continuous function, and let $\pi_\alpha : \prod_\alpha X_\alpha \to X_\alpha$ be the projection map. It is a trivial fact that $\pi_\alpha$ is continuous in both the product and box topologies. So, this means that

$$f_\alpha = \pi_\alpha \circ f$$

is a continuous map from $A$ to $X_\alpha$, because the composition of continuous maps is continuous. This proves the forward direction, and infact shows that the forward direction holds even in the box topology.

Conversely, suppose each $f_\alpha$ is a continuous map, and let $U$ be any basic open set in $\prod_\alpha X_\alpha$. So, we see that $U = \prod_\alpha U_\alpha$, where all but finitely many $\alpha$ satisfy $U_\alpha = X_\alpha$. Let $\alpha_1, ..., \alpha_n$ be the finitely many indices which don't satisfy $U_\alpha = X_\alpha$. It is easy to see that

$$f^{-1}(U) = \bigcap_{1 \leq i \leq n} f_{\alpha_i}^{-1}(U_{\alpha_i})$$

Now, each of the sets $f_{\alpha_i}^{-1}(U_{\alpha_i})$ is an open set in $A$ by the assumption. Since a finite intersection of open sets is open, it follows that $f^{-1}(U)$ is open in $A$, and this is precisely where the box topology won't work. This completes the proof. ∎

**Example 1.24.** We will now give a concrete example where each coordinate function is continuous, but the given function is not continuous in the box topology. Let $\mathbb{R}^\omega$ denote the product of countably infinite many copies of $\mathbb{R}$ (or simply the set of all sequences in $\mathbb{R}$), and let this space be equipped with the box topology. Consider the map

$$f : \mathbb{R} \to \mathbb{R}^\omega$$

given by $t \mapsto (t, t, t, ...)$. It is clear that each coordinate function is continuous, being the identity function from $\mathbb{R}$ to itself. But, we claim that $f$ is *not* continuous. To show this, let

$$U = (-1, 1) \times \left(\frac{-1}{2}, \frac{1}{2}\right) \times \left(\frac{-1}{3}, \frac{1}{3}\right) \times ... \subseteq \mathbb{R}^\omega$$

Clearly, $U$ is open in the box topology, since its a basic open set. Also, note that $0 \in f^{-1}(U)$. Infact, it is clearly seen that $f^{-1}(U) = \{0\}$, which is not open in $\mathbb{R}$.

## 1.7. Connectedness.

Let us now look at the notion of *connectedness*.

**Definition 1.10.** Let $X$ be a topological space. A *separation* of $X$ is a pair $U, V$ of open non-empty disjoint subsets of $X$ such that $X = U \cup V$. $X$ is said to be *connected* if it has no separation.

**Example 1.25.** Here are some trivial examples.

1. If $X$ is discrete and has atleast 2 elements, then $X$ is not connected.
2. The subspace $[-1, 0) \cup (0, 1]$ of $\mathbb{R}$ is not connected.
3. The subspace $\mathbb{Q}$ of $\mathbb{R}$ is not connected. Just take consider the sets $(\sqrt{2}, \infty) \cap \mathbb{Q}$ and $(-\infty, \sqrt{2}) \cap \mathbb{Q}$. A similar argument shows that $\mathbb{R} \setminus \mathbb{Q}$ is not connected.

We will now prove an alternate characterisation of connectedness, but we will need an easy lemma to do so.

**Lemma 1.8.** *Let $A \subseteq Y \subseteq X$, where $X$ is any topological space. Then, the closure of $A$ in $Y$ is equal to $\overline{A} \cap Y$, where $\overline{A}$ is the closure of $A$ in $X$.*

*Proof.* This is trivial, because $\overline{A} \cap Y$ is the smallest closed set in $Y$ that contains $A$. ■

**Proposition 1.9.** *Let $X$ be a topological space and let $Y \subseteq X$ be a subspace. A pair $A, B$ of subsets of $Y$ is a separation of $Y$ if and only if $A \cup B = Y$, $A, B$ are disjoint and neither contains a limit point of the other.*

*Proof.* Let $A, B$ be a separation of $Y$. Then, $A$ and $B$ are clearly both open and closed in $Y$. So, we see by **Lemma** 1.8 that $A = \overline{A} \cap Y$, and hence $\overline{A} \cap B = \phi$. This shows that $B$ doesn't contain any limit point of $A$ and vice-versa.

Conversely, suppose $A, B$ are disjoint subsets of $Y$ such that $A \cup B = Y$ and neither contains a limit point of the other. To show that $A, B$ is a separation of $Y$, it is enough to show that $A, B$ are open in $Y$.

We know that $\overline{A} \cap B = \phi$. Hence, $\overline{A} \cap Y = A$, i.e $A$ is closed in $Y$. Similarly, $B$ is closed in $Y$, and hence we are done. ■

**Lemma 1.10.** *If two subspaces $A, B$ form a separation of $X$ and $Y \subseteq X$ is a connected subspace, then $Y$ is contained entirely in $A$ or entirely in $B$.*

*Proof.* We have that $Y = (Y \cap A) \cup (Y \cap B)$, and both $Y \cap A$ and $Y \cap B$ are disjoint open subsets of $Y$. Since $Y$ is connected, one of the above sets must be empty, and this proves the claim. ■

**Theorem 1.11.** *Let $X$ be a topological space, and let $\{Y_\alpha\}$ be a collection of connected subspaces of $X$ such that $\bigcap_\alpha Y_\alpha \neq \phi$. Then, $\bigcup_\alpha Y_\alpha$ is a connected space.*

*Proof.* For the sake of contradiction, let $Y = A \cup B$ be a separation of $Y$, where $Y = \bigcup_\alpha Y_\alpha$. Now, note that each $Y_\alpha$ is connected, and hence by **Lemma** 1.10 we see that $Y_\alpha \subseteq A$ or $Y_\alpha \subseteq B$ for all $\alpha$. Now, we claim that either *all* $Y_\alpha$ lie in $A$, or *all* $Y_\alpha$ lie in $B$. But this is clear, because their intersection is non-empty, and hence they must all lie in one of $A$ or $B$. Without loss of generality, suppose $Y_\alpha \subseteq A$ for all $\alpha$. But, this implies that $B = \phi$, contradicting the fact that $A, B$ is a separation. This completes the proof. ■

**Remark 1.11.1.** The above proof actually works with a weaker hypothesis: $Y_\alpha \cap Y_\beta \neq \phi$ for all $\alpha, \beta$.

**Theorem 1.12.** *Let $A \subseteq X$ be a connected space, where $X$ is not necessarily connected. If $A \subseteq B \subseteq \overline{A}$, then $B$ is also connected. In simple words, if we have connected set, then adjoining some or all of its limit points still results in a connected set.*

*Proof.* For the sake of contradiction, suppose $B = C \cup D$ is a separation of $B$. Because $A$ is connected, we see that $A \subseteq C$ or $A \subseteq D$ by **Lemma** 1.10. Without loss of generality suppose $A \subseteq C$. Since $D$ is non-empty and $A \subseteq C$, we see that $D$ contains a limit point of $A$, because $\overline{A} = A \cup A'$. But, this clearly a contradiction, because $D$ is an open set containing a limit point of $A$, and hence it will intersect with $A$, i.e it will intersect with $C$. So, it follows that $B$ is connected. ■

**Theorem 1.13.** *The image of a connected set under a continuous function is connected.*

*Proof.* Let $f : X \to Y$ be a continuous map, where $Y = f(X)$. For the sake of contradiction, suppose $Y = A \cup B$ is a separation of $Y$. Clearly, it follows that $f^{-1}(A) \cup f^{-1}(B)$ is a separation of $X$, but this is a contradiction to the connectedness of $X$. So, $Y$ is connected. ∎

**Theorem 1.14.** *Let $\{X_\alpha\}$ be a family of connected topological spaces. Then $\prod_\alpha X_\alpha$ is also connected, where this space is given the product topology.*

*Proof.* We deal with two cases; the first case will be when we have a finite product, and the second is where we have an arbitrary product.

(1) *Finite products.* Without loss of generality, we can assume that the product has only two factors, and we can then induct on the number of factors. This is true because of the obvious fact that

$$(X \times Y) \times Z \cong X \times Y \times Z$$

So, let $X, Y$ be any two non-empty connected spaces, and we wish to show that $X \times Y$ is connected. Suppose $(a, b) \in X \times Y$ is fixed. Then we easily see that

$$X \cong X \times \{b\} \quad , \quad Y \cong \{x\} \times Y, \quad \forall x \in X$$

Now for any $x \in X$, note that $x \in X \times \{b\} \cap \{x\} \times Y$, and hence by **Theorem** 1.11 we see that $X \times \{b\} \cup \{x\} \times Y$ is connected for all $x \in X$. Finally, note that

$$X \times Y = \bigcup_{x \in X} (X \times \{b\} \cup \{x\} \times Y)$$

is connected by another application of **Theorem** 1.11, because $(a, b) \in X \times \{b\} \cup \{x\} \times Y$ for all $x \in X$. So, it follows that $X \times Y$ is a connected set.

(2) Next, we will deal with the case of arbitrary products. So, let $\{X_\alpha\}$ be a family of non-empty connected topological spaces. Let $(b_\alpha) \in \prod_\alpha X_\alpha = X$ be any point (such a point exists by invoking the Axiom of Choice). Now, given a finite set $\{\alpha_1, ..., \alpha_n\} \subseteq J$, define

$$X_{\alpha_1,...,\alpha_n} = \{(x_\alpha) \in X \mid x_\alpha = b_\alpha \ \forall \ \alpha \neq \alpha_1, ..., \alpha_n\}$$

It is then easily seen that

$$X_{\alpha_1,...,\alpha_n} \cong X_{\alpha_1} \times ... \times X_{\alpha_n}$$

and hence each $X_{\alpha_1,...,\alpha_n}$ is a connected set. Now note that if $\{\alpha_1, ..., \alpha_n\} \subseteq J$ is any finite subset, then

$$(b_\alpha) \in X_{\alpha_1,...,\alpha_n}$$

So by **Theorem** 1.11, we see that

$$Y = \bigcup_{\{\alpha_1,...,\alpha_n\} \subseteq J} X_{\alpha_1,...,\alpha_n}$$

is a connected set, where the union is taken over all finite subsets of $J$. Note that we are not done yet, because $X \neq Y$ in general. However, we show that

$$X = \overline{Y}$$

To show this, we will show that any point of $X$ is either in $Y$ or is a limit point of $Y$. So, let $(a_\alpha) \in X$. If $(a_\alpha) \in Y$, then we are done, and hence we assume $(a_\alpha) \notin Y$. So, we need to show that $(a_\alpha)$ is a limit point of $Y$. Let $\bigcup_\alpha U_\alpha = U$ be any basic open set containing the point $(a_\alpha)$. Since we are in the product topology, this means that $U_\alpha = X_\alpha$ for all but finitely many $\alpha$, and this is where the box topology won't work. We will show that

$$U \cap Y \neq \phi$$

Let $\alpha_1, ..., \alpha_n \in J$ be those indices for which $U_\alpha = X_\alpha$ does not hold. Define the point $(y_\alpha) \in X$ by

$$y_\alpha = \begin{cases} a_\alpha & , \quad \alpha = \alpha_1, ..., \alpha_n \\ b_\alpha & , \quad \text{otherwise} \end{cases}$$

Then it is easily seen that $(y_\alpha) \in Y \cap U$. So, it follows that $X = \overline{Y}$ and hence by **Theorem** 1.12 we see that $X$ is connected.

■

**Example 1.26.** In this example, we will show that $\mathbb{R}^\omega$ is not connected in the box topology. This will show that the product topology is really needed in the statement of **Theorem** 1.14. We will be assuming that $\mathbb{R}$ is connected, and this we will show in the next section.

Let

$$U = \{(a_n) \in \mathbb{R}^\omega \mid (a_n) \text{ is a bounded sequence}\} \neq \mathbb{R}^\omega$$

and we have that $U \neq \phi$. We claim that $U$ is both open and closed in the box topology. Let $a = (a_n) \in \mathbb{R}^\omega$. Define

$$W_a = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times ...$$

Clearly, $W_a$ is a basic open set in the box topology in $\mathbb{R}^\omega$. Moreover, we see that if $a \in U$, then $W_a \subseteq U$ and that if $a \notin U$ then $W_a \subseteq \mathbb{R}^\omega \setminus U$. So, it follows that both $U$ and $U^c$ are open, i.e $U$ is a non-empty open and closed subset of $\mathbb{R}^\omega$, and hence $\mathbb{R}^\omega$ is not connected.

1.8. **A generalisation of reals being connected.** In this section, our main goal will be to prove that $\mathbb{R}$ is connected. We will actually prove a more general result, and conclude the connectedness of $\mathbb{R}$ from this.

**Definition 1.11.** Let $X$ be an ordered set. $X$ is said to have the *least upper bound property* if every non-empty, bounded above subset of $X$ has a least upper bound in $X$.

**Definition 1.12.** An ordered set $X$ is said to be a *linear continuum* if $X$ has the least upper bound property and if $x < y$ are in $X$, then there is some $z \in X$ with $x < z < y$.

**Theorem 1.15.** *Let $L$ be a linear continuum in the order topology. Then $L$, every interval in $L$ and every ray in $L$ are all connected.*

**Remark 1.15.1.** An *interval* in $L$ is a set of the form $(\alpha, \beta)$, $[\alpha, \beta]$, $[\alpha, \beta)$ or $(\alpha, \beta]$, where $\alpha, \beta \in L$. A *ray* is an interval which is unbounded in atleast one direction.

*Proof.* Let $Y \subseteq L$ be a subspace of $L$, which is either $L$, an interval in $L$ or a ray in $L$.

Note that $Y$ is *convex*, i.e if $a < b \in Y$ then $[a, b] \subseteq Y$. For the sake of contradiction, suppose $Y$ is not connected. Let $Y = A \cup B$ be a separation of $Y$. Let $a \in A$ and $b \in B$. Then, $[a, b] = A_0 \cup B_0$ where $A_0 = A \cap [a, b]$ and $B_0 = B \cap [a, b]$. Note that $A_0, B_0$ are disjoint, non-empty open subsets of $[a, b]$.

Note that $A_0$ is bounded above in $Y$, by $b$ for example. So let $c = \sup A_0 \in L$. But infact, $c \in [a, b]$ because $c \geq a$ and $c \leq b$. We claim that $c \notin A_0 \cup B_0$.

First, we show that $c \notin A_0$. For the sake of contradiction, suppose $c \in A_0$. Then $c \neq b$ and infact $c < b$. So $c = a$ or $a < c < b$. In either case, we have an interval of the form $[c, e) \subseteq A_0$. This is because $A_0$ is open in $[a, b]$. Let $z \in L$ be such that $c < z < e$, and this is true by the linear continuum property. Hence $z \in A_0$. But this contradicts the fact that $c = \sup A_0$. So, it follows that $c \notin A_0$.

Next, we show that $c \notin B_0$. To get a contradiction, suppose $c \in B_0$. Then $a \neq c$, and hence either $c = b$ or $a < c < b$. In either case, we see that there is an interval $(d, c] \subseteq B_0$, and this is true because $B_0$ is open in $[a, b]$. So, we see that $d$ is an upper bound of $A_0$, which again contradicts the fact that $c = \sup A_0$. So, $c \notin B_0$.

All of this means that $c \notin A_0 \cup B_0$, which is clearly a contradiction because $c \in [a, b]$. Hence, it follows that $Y$ must be connected. ■

**Remark 1.15.2.** The proof shows that any convex subset of a linear continuum is connected.

**Corollary 1.15.1.** *$\mathbb{R}$ in the standard topology is connected, because the standard topology of $\mathbb{R}$ is equivalent to the order topology of $\mathbb{R}$.*

**Corollary 1.15.2.** *$I \times I$ in the dictionary order topology is connected. $I \times I$ is also connected in the standard topology.*

**Theorem 1.16** (**Intermediate Value Theorem**)**.** *Let $f : X \to Y$ be a continuous function with $X$ connected and $Y$ ordered (and given the order topology). If $a, b \in X$ and $\gamma \in Y$ is such that $f(a) \leq \gamma \leq f(b)$, then there is some $c \in X$ such that $f(c) = \gamma$.*

*Proof.* This is an easy connectedness argument. First, observe that $f(X)$ is connected in the order topology on $Y$. To get a contradiction, suppose there is no $c \in X$ such that $f(c) = \gamma$, i.e $\gamma \notin f(X)$. Now, we can write

$$f(X) = \{y \in f(X) \mid y < \gamma\} \cup \{y \in f(X) \mid y > \gamma\}$$

and it is easy to see that this form a separation of $f(X)$, because each of the two sets in the above union is non-empty and open in $f(X)$.

It must be noted that this theorem has nothing to do with **Theorem** 1.15. ■

**Corollary 1.16.1.** *Since $[0, 1]$ is connected, we get the usual intermediate value theorem.*

1.9. **Path Connectedness.** Here we see another important topological property, which is a special type of connectedness.

**Definition 1.13.** Let $X$ be a topological space and $x, y \in X$. A *path* from $x$ to $y$ is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

**Remark 1.16.1.** The interval $[0, 1]$ can be replaced by any closed interval $[a, b]$ in $\mathbb{R}$ with $a < b$, because $[a, b] \cong [0, 1]$.

**Definition 1.14.** A topological space $X$ is said to be *path-connected* if there is a path between any two points of $X$.

**Proposition 1.17.** *If $X$ is path connected, then $X$ is connected. The converse is not true in general (see* **Example** 1.28 *for a counterexample).*

*Proof.* Suppose $X$ is path connected, and for the sake of contradiction suppose $X = A \cup B$ is a separation of $X$. Let $a \in A$ and $b \in B$ be any two points, and let $f$ be a path from $a$ to $b$. Since $f$ is continuous, the image $f([0, 1])$ is a subset of $X$. However, note that $f([0, 1])$ intersects with both $A$ and $B$, and this is clearly a contradiction to **Lemma** 1.10. So, it follows that $X$ is connected. ■

**Example 1.27.** The unit ball $B^n := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1^2 + ... + x_n^2 = 1\}$ is path connected, and infact the straight line path between any two points suffices.

**Example 1.28.** Let $X = I \times I$ in the dictionary order topology. We know that $X$ is connected by **Theorem** 1.15. We will show that $X$ is *not* path-connected. Let $p = (0, 0)$ and $q = (1, 1)$. Let $f : [0, 1] \to I \times I$ be a continuous map such that $f(0) = p$ and $f(1) = q$. Then $f([0, 1]) = I \times I$ by the **Intermediate Value Theorem** 1.16. Now for every $x \in I$, let $U_x := f^{-1}(\{x\} \times (0, 1)) \subseteq [0, 1]$. Then $U_x$ is a non-empty open subset of $[0, 1]$. Now, there is some $q_x \in \mathbb{Q} \cap U_x$. Consider the map $g : [0, 1] \to \mathbb{Q}$ given by $g(x) = q_x$. Clearly, $g$ is an injective map, because the $U_x$'s are disjoint. But $[0, 1]$ is uncountable, and hence this is a contradiction. So, $X$ is *not* path connected.

**Example 1.29.** Here we will see the so called *topologist's sine curve*. Define

$$S := \left\{ \left( x, \sin \frac{1}{x} \right) \mid x \in (0, 1] \right\} \subseteq \mathbb{R}^2$$

Clearly, $S$ is connected being the image of $(0, 1]$ under a continuous function. Let $\overline{S}$ be the closure of $S$ in $\mathbb{R}^2$. We see that

$$\overline{S} = 0 \times [-1, 1] \cup S$$

$\overline{S}$ is also connected being the closure of a connected set. We show that $\overline{S}$ is *not* path connected. We claim that there is no path from a point in $0 \times [-1, 1]$ to a point in $S$. Let $f : [0, 1] \to \overline{S}$ be a path such that $f(0) \in 0 \times [-1, 1]$ and $f(1) \in S$. Put

$$A := \{t \in [0, 1] \mid f(t) \in 0 \times [-1, 1]\} \subseteq [0, 1]$$

Then $A$ is a proper, non-empty closed subset of $[0, 1]$ (proper because $1 \notin A$ and non-empty because $0 \in A$). Let $a = \sup A \in [0, 1]$. So

$$f|_{[a,1]} : [a, 1] \to \overline{S}$$

is a continuous function such that $f(a) \in 0 \times [-1, 1]$ and $f((a, 1]) \subseteq S$. Since $[a, 1] \cong [0, 1]$, we get a continuous function $f : [0, 1] \to \overline{S}$ such that $f(0) \in 0 \times [-1, 1]$ and $f((0, 1]) \subseteq S$. Let $f(t) = (x(t), y(t))$ where $x, y : [0, 1] \to \mathbb{R}$ such that $x(0) = 0$, $y(0) \in [-1, 1]$ and $x(t) > 0$ for all $t > 0$ and $y(t) = \sin\left(\frac{1}{x(t)}\right)$ for all $t > 0$.

Let $n \geq 1$. Choose $u$ such that $0 < u < x\left(\frac{1}{n}\right)$ and $\sin\left(\frac{1}{u}\right) = (-1)^n$. By the **Intermediate Value Theorem** there is some $t_n$ such that $0 < t_n < \frac{1}{n}$ such that $x(t_n) = u$. Then $y(t_n) = \sin\left(\frac{1}{x(t_n)}\right) = (-1)^n$ So, $t_n \in [0, 1]$ and $t_n \to 0$ but $y(t_n) = (-1)^n$ diverges, and this is a contradiction since $y$ is a continuous function. So, the sine curve is connected but *not* path connected.

**Example 1.30.** Our next example will be the *comb space*. Let $K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and put

$$C := [0, 1] \times \{0\} \cup K \times [0, 1] \cup \{0\} \times [0, 1]$$

and let

$$D := C - \{0\} \times (0, 1)$$

and let $p = (0, 1) \in D$, and note that $(0, 0) \in D$. We claim that $D$ is connected but *not* path connected. $C$ is called the *comb space* and $D$ is called the *deleted comb space*.

Note that $[0, 1] \times \{0\} \cup K \times [0, 1]$ is path connected, and hence it is connected. So $D$ is connected because $p$ is a limit point of $[0, 1] \times \{0\} \cup K \times [0, 1]$. Now we show that $D$ is *not* path connected. So suppose $f : [0, 1] \to D$ is a continuous function such that $f(0) = p$.

We will show that $f([0,1]) = \{p\}$. Since $\{p\} \subseteq D$ is closed, $f^{-1}(p)$ is closed in $[0,1]$. If we show that $f^{-1}(p)$ is also open in $[0,1]$ then we are done. Let $V \subseteq \mathbb{R}^2$ be an open set such that $p \in V$ and $V$ does not intersect the $x$-axis. Clearly, $0 \in f^{-1}(V)$ and $f^{-1}(V)$ is open in $[0,1]$. Let $U$ be a basic open set in $[0,1]$ such that $0 \in U$ and $U \subseteq f^{-1}(V)$. So $U$ is connected, and hence $f(U)$ is connected. Suppose $f(U) \neq \{p\}$, and let $q \in f(U)$ with $q \neq p$. Since $q$ is not on the $x$-axis, we have $q = \frac{1}{n} \times t_0$ where $n \geq 1$ is an integer and $t_0 \in [0,1]$.

Now choose $r$ such that $\frac{1}{n+1} < r < \frac{1}{n}$. The sets $(-\infty, r) \times \mathbb{R}$ and $(r, \infty) \times \mathbb{R}$ cover $D$. Since $f(U)$ is connected, we see that $f(U) \subseteq (-\infty, r) \times \mathbb{R}$ or $f(U) \subseteq (r, \infty) \times \mathbb{R}$. This is a contradiction, because $p, q \in f(U)$.

So, $f(U) = \{p\}$ and $f^{-1}(p)$ is open in $[0,1]$.

**Lemma 1.18.** *No two of the spaces* $(0,1)$*,* $[0,1)$ *and* $[0,1]$ *are homeomorphic.*

*Proof.* If there is a homeomorphism $f : [0,1) \to (0,1)$ then we also have a homeomorphism $f|_{(0,1)} : (0,1) \to (0,1) \setminus f(0)$, which is a contradiction because $(0,1)$ is connected and $(0,1) \setminus f(0)$ is not connected. ■

1.10. **Compactness.** Here we will see a generalisation of compactness in topological spaces.

**Definition 1.15.** Let $X$ be a space. A collection of subsets of $X$ is a *covering* or a *cover* of $X$ if their union is $X$. An *open covering* is one in which each subset is open.

**Definition 1.16.** A space $X$ is *compact* if every open cover of $X$ contains a finite subcover. This is the usual definition of compactness that we saw in metric spaces.

**Definition 1.17.** Let $X$ be a topological space and $Y \subseteq X$. A *covering* of $Y$ by open sets in $X$ is a collection of open sets in $X$ such that $Y$ is a subset of the union of the collection.

**Lemma 1.19.** *Let $X$ and $Y$ be such that $Y \subseteq X$. Then $Y$ is compact if and only if every covering of $Y$ by open sets in $X$ has a finite subcover.*

*Proof.* First, suppose $Y$ is compact. So, any open cover of $Y$ (note, by an open cover, we mean a cover using sets open *in* $Y$) has a finite subcover. Now, let $\{U_\alpha\}$ be a collection of open sets in $X$ which cover $Y$. So, we immediately see that

$$\bigcup_\alpha (U_\alpha \cap Y) = Y$$

and note that each $U_\alpha \cap Y$ is open in $Y$. By our assumption, there are $\alpha_1, ..., \alpha_n$ such that

$$\bigcup_{i=1}^n (U_{\alpha_i} \cap Y) = Y$$

and hence it follows that

$$Y \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

showing the existence of a finite subcover. The converse is also similarly proven. ■

**Proposition 1.20.** *If $X$ is compact and $Y \subseteq X$ is closed, then $Y$ is also compact. In simple words, closed subsets of compact sets is compact.*

*Proof.* Let $\mathcal{O}$ be a covering of $Y$ by open sets in $X$. Then $\mathcal{O} \cup \{X \setminus Y\}$ is an open cover of $X$. Since $X$ is compact, there is a finite subcover. Discarding $X \setminus Y$, if needed, we obtain a finite subcover of $Y$. ∎

**Proposition 1.21.** *If $X$ is Hausdorff and $Y \subseteq X$ is compact then $Y$ is closed in $X$.*

*Proof.* We show that $X - Y$ is open in $X$. Let $x \in X - Y$. For each $y \in Y$, let $U_y$, $V_y$ be disjoint open sets such that $x \in U_y$ and $y \in V_y$. So $\{V_y \mid y \in Y\}$ is an open cover of $Y$ by open sets in $X$. Because $Y$ is compact, there are $y_1, ..., y_n \in Y$ such that $Y \subseteq V_{y_1} \cup ... \cup V_{y_n}$. Now let $U = U_{y_1} \cap U_{y_2} \cap ... \cap U_{y_n}$. Clearly, $U$ is an open neighborhood of $x$. Clearly, $U \cap Y = \phi$, so that $U \subseteq X - Y$. This completes the proof. ∎

**Lemma 1.22.** *Let $X$ be Hausdorff, and let $Y \subseteq X$ be compact. For any $x \in X - Y$, there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $Y \subseteq V$. So in simple words, it is possible in a Hausdorff space to separate a compact set from a point in its compliment.*

*Proof.* The proof is exactly the same as in the previous proposition. Just take $V = V_{y_1} \cup ... \cup V_{y_n}$ and $U = U_{y_1} \cap ... \cap U_{y_n}$. ∎

**Example 1.31.** By **Proposition** 1.21 we immediately see that the sets $(a, b)$, $[a, b)$ and $(a, b]$ are not compact in $\mathbb{R}$, because $\mathbb{R}$ is Hausdorff.

**Example 1.32.** Let $X$ be any set with the finite complement topology. We show that any subset of $X$ is compact. Let $A \subseteq X$ be any subset, and let $\{U_\alpha\}$ be a covering of $A$ by open subsets of $X$. Let $U_\alpha$ be any of these sets in the open cover. If $A \subseteq U_\alpha$, then we are done. So, suppose $A \cap U_\alpha^c \neq \phi$. But, we know that $U_\alpha^c$ is finite, and hence $A \cap U_\alpha^c$ contains finitely many points. So we see that $A \subseteq U_\alpha \cup U_{\alpha_1} \cup ... \cup U_{\alpha_n}$ for some $\alpha_1, ..., \alpha_n$, and hence we have extracted a finite subcover.

**Theorem 1.23.** *Let $f : X \to Y$ be a continuous map. If $X$ is compact, then $f(X)$ is also compact.*

*Proof.* This is a simple proof which we have done many times before. ∎

**Corollary 1.23.1.** *If $X$ and $Y$ are homeomorphic, then $X$ is compact if and only if $Y$ is compact.*

1.11. **Finite Products of Compact Spaces.** In this section, we will study finite products of compact spaces, and see whether they are compact.

**Proposition 1.24** (**Tube Lemma**)**.** *Let $X, Y$ be topological spaces, and suppose $Y$ is compact. Let $N \subseteq X \times Y$ be an open set such that $x_0 \times Y \subseteq N$ for some $x_0 \in X$. Then there is an open set $W \subseteq X$ such that $x_0 \in W$ and $W \times Y \subseteq N$.*

**Remark 1.24.1.** The set $W \times Y$ is called a *tube*. This proposition basically says that if $Y$ is compact and if a vertical line sits inside an open subset of $X \times Y$, then infact a tube sits inside that open subset.

*Proof.* Because $Y$ is compact, we know that $x_0 \times Y$ is compact, because these are homeomorphic spaces.

For every $y \in Y$, there is a basic open set $U \times Y$ such that $(x_0, y) \in U \times V \subseteq N$. So we can cover $x_0 \times Y$ by finitely many basic open sets of the form $U \times V$, each contained in $N$. In other words, there are open sets $U_1, U_2, ..., U_n \subseteq X$ and $V_1, V_2, ..., V_n \subseteq Y$ such that

$$x_0 \times Y \subseteq (U_1 \times V_1) \cup ... \cup (U_n \times V_n) \quad , \quad U_i \times V_i \subseteq N \ \forall 1 \leq i \leq n$$

Let $W = U_1 \cap U_2 \cap ... \cap U_n$. Clearly, $x_0 \in W$. We show that $W \times Y \subseteq N$. To see this, suppose $(x, y) \in W \times Y$. Then $(x_0, y) \in U_i \times V_i$ for some $1\ ei \leq n$. Then $(x, y) \in U_i \times V_i$, and hence $(x, y) \in N$. This shows that $W \times Y \subseteq N$, and this completes the proof. ∎

**Theorem 1.25.** *Any finite product of compact spaces is compact.*

*Proof.* Clearly, it suffices to show this for the product of two compact spaces, since the argument can then be extended via induction.

So, let $X, Y$ be compact spaces, and we will show that $X \times Y$ is a compact space. Let $\mathcal{O}$ be an open cover of $X \times Y$. Let $x_0 \in X$. Now, observe that $x_0 \times Y$ is also a compact set; hence, a finite subcollection of $\mathcal{O}$ covers $x_0 \times Y$, i.e there are $A_1, ..., A_n \in \mathcal{O}$ such that

$$x_0 \times Y \subseteq A_1 \cup A_2 \cup ... \cup A_n =: N$$

By the **Tube Lemma** 1.24, there is an open set $W \subseteq X$ containing $x_0$ such that

$$W \times Y \subseteq N$$

which means that $W \times Y$ is covered by $A_1, ..., A_n$.

What we have shown is this: for all $x \in X$, there is an open set $W_x \subseteq X$ containing $x$ such that $W_x \times Y$ is covered by finitely many elements of $\mathcal{O}$. Now, the collection $\{W_x\}_{x \in X}$ is an open cover of $X$ and because $X$ is compact, there are $x_1, ..., x_m \in X$ such that $X = W_{x_1} \cup ... \cup W_{x_m}$. So, it follows that finitely many elements of $\mathcal{O}$ covers $X \times Y$, and this completes the proof. ∎

**Exercise 1.8.** Find an example where the **Tube Lemma** 1.24 fails.

**Solution.** Let $X = Y = \mathbb{R}$, and so we are considering the space $\mathbb{R}^2$. Just take any open subset of $\mathbb{R}^2$ whose *width* gets infinitesimally small; for instance, take the open region in the first quadrant bounded by the graph of the function $f(x) = 1/x$ and its reflection around the $y$-axis.

1.12. **Compact subsets of ordered sets with the LUB property.** First, we look at an alternate characterisation of compactness.

**Definition 1.18.** Let $X$ be a topological space. A collection $\mathcal{C}$ of closed sets in $X$ is said to have the *finite intersection property* if for any finite number of sets $C_1, ..., C_n \in \mathcal{C}$, it is true that $C_1 \cap C_2 \cap ... \cap C_n \neq \phi$.

**Theorem 1.26.** *Let $X$ be a topological space. Then, $X$ is compact if and only if for every collection $\mathcal{C}$ of closed sets in $X$ having the finite intersection property, it is true that*

$$\bigcap_{C \in \mathcal{C}} C \neq \phi$$

*Proof.* First, suppose $X$ is compact, and let $\mathcal{C}$ be any collection of closed subsets of $X$ having the finite intersection property. For the sake of contradiction, suppose

$$\bigcup_{C \in \mathcal{C}} C = \phi$$

and taking complements, this means

$$\bigcap_{C \in \mathcal{C}} C^c = X$$

and hence we have an open cover of $X$. Because $X$ is compact, there are finitely many $C_1, C_2, ..., C_n \in \mathcal{C}$ such that

$$C_1^c \cup ... \cup C_n^c = X$$

and taking complements, this implies that

$$C_1 \cap ... \cap C_n = \phi$$

which is a contradiction. This proves the forward direction.

Conversely, suppose for every collection $\mathcal{C}$ of closed subsets of $X$ having the finite intersection property, it is true that

$$\bigcap_{C \in \mathcal{C}} C \neq \phi$$

Let $\{U_\alpha\}$ be an open covering of $X$, i.e

$$\bigcup_\alpha U_\alpha = X$$

Taking complements, we see that

$$\bigcap_\alpha U_\alpha^c = \phi$$

and hence by our assumption, it follows that $\{U_\alpha^c\}$ cannot be a collection of closed sets having the finite intersection property. This means that there are $\alpha_1, ..., \alpha_n$ such that

$$U_{\alpha_1}^c \cap ... \cap U_{\alpha_n}^c = \phi$$

and taking complements, we have obtained a finite subcover, implying that $X$ is compact. This completes the proof. ■

**Theorem 1.27.** *Let $X$ be an ordered topological space satisfying the least upper bound property. Then every closed interval in $X$ is compact.*

*Proof.* Let $a, b \in X$ such that $a < b$. We show that $[a, b]$ is compact. Let $\mathcal{O}$ be an open cover of $[a, b]$ by open sets in $X$. We will prove the claim in a series of steps.

(1) We show that for each $x \in [a, b]$ with $x \neq b$, there is some $y \in [a, b]$ such that $y > x$ and $[x, y]$ is covered by atmost 2 elements of $\mathcal{O}$. If $x$ has an immediate successor, say $y \in X$ then we take $[x, y] = \{x, y\}$ and atmost two elements of $\mathcal{O}$ cover $[x, y]$. If $x$ has no immediate successor, let $A \in \mathcal{O}$ be an open set containing $x$. Then there is some $c \in [a, b]$ such that $x \in [x, c) \subseteq A$ such that there is some $y \in [x, c)$ with $y > x$ (note that $x \neq b$ is required for this). So, we can just consider the interval $[x, y]$.

(2) Let

$$C := \{y \in (a, b] \mid [a, y] \text{ can be covered by finitely many elements of } \mathcal{O}\}$$

By step (1), we see that $C$ is non-empty. Let $c = \sup C \in [a, b]$.

(3) We show that $c \in C$. Let $A \in \mathcal{O}$ such that $c \in A$. Then there is some $d \in [a, b]$ such that $c \in (d, c] \subseteq A$, i.e we know that $d < c$. Now, if $C \cap (d, c] = \phi$, then this would imply that $d$ is an upper bound of $C$, which is a contradiction. So, it follows that $C \cap (d, c] \neq \phi$, and let $z \in C \cap (d, c]$. Since $z \in C$, the set $[a, z]$ is covered by finitely many elements of $\mathcal{O}$. But then, it follows that $[a, c]$ is also covered by finitely many elements of $\mathcal{O}$, by simply adding the set $A$. Hence, we see that $c \in C$.

(4) Finally, we show that $c = b$. If $c < b$, then step (1) applied to $c$ will show that there is some $y \in [a, b]$ with $y > c$ such that $[c, y]$ is covered by atmost two elements of $\mathcal{O}$. This means that $y \in C$, which contradicts the fact that $c = \sup C$.

Step (4) implies that $[a, b]$ can be covered by finitely many elements of $\mathcal{O}$, and this completes our proof. ■

**Corollary 1.27.1.** *Every closed interval in $\mathbb{R}$ is compact.*

**Theorem 1.28** (**Heine-Borel**)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded (in the Euclidean or the square metric).*

*Proof.* Let $\rho$ denote the square metric. Suppose $A \subseteq \mathbb{R}^n$ is compact. Since $\mathbb{R}^n$ is Hausdorff, $A$ is closed. Cover $A$ by $B(a, n)$ for a fixed $a \in \mathbb{R}^n$ and vary $n \in \mathbb{N}$. This shows that $A$ is bounded in $\mathbb{R}^n$.

Conversely, suppose $A$ is closed and bounded in $\mathbb{R}^n$. So, there is some $N \in \mathbb{N}$ such that

$$A \subseteq \overline{B_\rho(0, N)} = [-N, N]^n$$

We know that $[-N, N] \subseteq \mathbb{R}$ is compact, and that a finite product of compact spaces is compact by **Theorem** 1.25, so it follows that $[-N, N]^n$ is compact. So, $A$ is closed in $[-N, N]^n$, and hence $A$ is compact. ■

**Example 1.33.** The sets $S^{n-1}$ , $B^n$ (the closed ball) are all compact in $\mathbb{R}^n$ by the above theorem.

**Theorem 1.29.** *Let $f : X \to Y$ be a continuous function from a compact space $X$ to an ordered topological space $Y$. Then there are $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.*

**Remark 1.29.1.** This is a generalisation of the usual extreme value theorem.

*Proof.* We have to show that $A = f(X)$ has a maximum and a minimum element.

If $A$ has no maximum element, then the following is an open cover of $A$:

$$\{(-\infty, a) : a \in A\}$$

Because $A$ is compact, there are $a_1, a_2, ..., a_n \in A$ such that

$$A \subseteq (-\infty, a_1) \cup ... \cup (-\infty, a_n)$$

Let $a = \max\{a_1, ..., a_n\}$. Then $a \notin (\infty, a_j)$ for all $j$. But $a \in A$, and this is a contradiction. So, it follows that there must be some maximum element. A similar argument shows that there is some minumum element. ■

1.13. **Some Familiar Results.** In this section, we will topologically prove some results which are familiar in analysis.

**Definition 1.19.** Let $(X, d)$ be a metric space, and let $A \subseteq X$ be a non-empty subset. For $x \in X$, define

$$d(x, A) := \inf\{d(x, a) \mid a \in A\}$$

It can be shown that the map $x \mapsto d(x, A)$ is continuous.

**Definition 1.20.** Let $\phi \neq A \subseteq X$ be bounded. The *diameter* of $A$ is defined as

$$\text{diam}(A) := \sup\{d(a, b) \mid a, b \in A\}$$

**Proposition 1.30** (**Lebesgue Number Lemma**). *Let $(X, d)$ be a compact metric space. Let $\mathcal{O}$ be an open cover of $X$. Then there is some $\delta > 0$ such that for each subset $A \subseteq X$ of diameter $\leq \delta$, then there is some $U \in \mathcal{O}$ such that $A \subseteq U$.*

**Remark 1.30.1.** Such a $\delta$ is called a *Lebesgue number* for the cover $\mathcal{O}$.

*Proof.* Let $\mathcal{O}$ be an open cover of $X$. If $X \in \mathcal{O}$, then take $\delta$ to be any positive real number. So, we assume that $X \notin \mathcal{O}$. let $\{A_1, ..., A_n\}$ be a finite subcover of $X$, and let $C_i = X \setminus A_i$. So we see that $C_i \neq \phi$ for each $i$. Define $f : X \to \mathbb{R}$ given by

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$$

i.e $f(x)$ is the *average distance* of $x$ from the sets $C_i$. It is easily seen that $f$ is continuous, being a sum of continuous functions. We claim that

$$f(x) > 0 \quad , \quad \forall x \in X$$

To prove this, let $x \in X$ and let $x \in A_i$. Let $\epsilon > 0$ be such that $B_\epsilon(x) \subseteq A$ (possible because $A_i$ is open). Now, if $y \in C_i$, then $y \notin A_i$ and hence $y \notin B_\epsilon(x)$ which implies that $d(x, y) \geq \epsilon$. This implies that

$$d(x, C_i) \geq \epsilon \implies f(x) \geq \frac{\epsilon}{n} > 0$$

Let $\delta$ be the minimum value of $f$, which exists by **Theorem** 1.29 ($X$ is compact), and clearly $\delta > 0$. We claim that $\delta$ is a Lebesgue number for the open cover $\mathcal{O}$. To show this, let $B \subseteq X$ be a subset such that $\operatorname{diam}(B) \leq \delta$. Let $x_0 \in B$. So, we see that $B \subseteq B_\delta(x_0)$. Let $d(x_0, C_m) = \max_i\{d(x_0, C_i)\}$. Then, we see that

$$\delta \leq f(x_0) = \frac{1}{n} \sum_i d(x_0, C_i) \leq d(x_0, C_m)$$

and hence

$$B_\delta(x_0) \subseteq X \setminus C_m = A_m$$

and hence we see that $B \subseteq A_m$. ∎

**Theorem 1.31.** *Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous map with $X$ compact. Then $f$ is uniformly continuous.*

*Proof.* Let $\epsilon > 0$. Cover $Y$ by the sets $\{B(y, \epsilon/2) \mid y \in Y\}$. Let

$$\mathcal{O} := \text{the open cover given by } \{f^{-1}(B(y, \epsilon/2)) \mid y \in Y\}$$

Let $\delta$ be the Lebesgue number for $\mathcal{O}$ ($\delta > 0$, exists by the **Lebesgue Number Lemma** 1.30). Now let $x_0, x_1 \in X$ with $d_X(x_0, x_1) < \delta$. Then $\operatorname{diam}\{x_0, x_1\} < \delta$. Hence there is some $y \in Y$ such that

$$\{x_0, x_1\} \subseteq f^{-1}(B(y, \epsilon/2))$$

which implies that

$$\{f(x_0), f(x_1)\} \subseteq B(y, \epsilon/2)$$

which implies that

$$d_Y(f(x_0), f(x_1)) < \epsilon$$

∎

**Definition 1.21.** Let $X$ be a space. A point $x \in X$ is said to be an *isolated point* of $X$ if $\{x\}$ is open in $X$.

**Proposition 1.32.** *Let $X$ be a topological space. Then $X$ has no isolated points if and only if every point of $X$ is a limit point of $X$.*

*Proof.* First, suppose $X$ has no isolated points, and let $x$ be any point of $X$. Let $W$ be any open neighborhood of $x$. We need to show that $W \cap X \setminus \{x\} \neq \phi$. But this is clear, because $x$ is *not* an isolated point of $X$. This shows that $x$ is a limit point of $X$.

The converse is straightforward. This completes the proof. ■

**Theorem 1.33.** *Let $X$ be a nonempty compact Hausdorff space. If $X$ has no isolated points, then $X$ is uncountable.*

*Proof.* First, we show the following: given a non-empty open set $U \subseteq X$ and $x \in X$, there exists an open set $V \subseteq X$ such that $V \subseteq U$, $x \notin \overline{V}$ and $V \neq \phi$. To prove this, let $y \in U$ such that $y \neq x$ (such a point $y$ exists because $x$ is not isolated, in particular $U \neq \{x\}$). By Hausdorffness, there are open sets $W_1, W_2$ such that $x \in W_1, y \in W_2$ and $W_1 \cap W_2 = \phi$. Then we claim that the set $V := U \cap W_2$ does the job. Clearly, $V$ is an open set, and it is non-empty because it contains the point $y$. Also, $V \subseteq U$ is clear. Finally, observe that $x$ is not a limit point of $V$, because $W_1 \cap V = \phi$, and hence this means that $x \notin \overline{V}$. This proves the claim.

Now let $f : \mathbb{N} \to X$ be a function. Let $x_n = f(n)$. Start with $U = X$ and the point $x_1$. Applying the above claim, we see that there is some non-empty open set $V_1$ such that $V_1 \subseteq U$ and $x_1 \notin \overline{V_1}$. Continue with $V_1$ and $x_2$: there is some non-empty open set $V_2$ such that $V_2 \subseteq V_1$ and $x_2 \notin \overline{V_2}$. Continuing this way, we get a chain

$$\overline{V_1} \supseteq \overline{V_2} \supseteq \overline{V_3} \supseteq \cdots$$

We know that $\overline{V_i} \neq \phi$ for all $i$. So this collection of closed sets satisfied the finite intersection property. Since $X$ is compact, it follows from **Theorem** 1.26 that

$$\bigcap_i \overline{V_i} \neq \phi$$

So take any $x \in \bigcap_i \overline{V_i}$. But this means that $x \neq x_i$ for all $i \in \mathbb{N}$, which means that $f$ is *not* surjective. Hence, $X$ cannot be countable, and this completes our proof. ■

**Corollary 1.33.1.** $\mathbb{R}$ *is uncountable and* $(a, b)$ *with* $a < b$ *is uncountable.*

1.14. **Countability Axioms.** Throughout, let $X$ be a topological space.

**Definition 1.22.** Let $x \in X$. A collection of neighborhoods of $x$, say $\{B_\alpha\}$, is said to be a *basis of open sets at $x$* if every neighborhood of $x$ contains some $B_\alpha$.

**Definition 1.23.** Here we define some countability axioms.

(1) $X$ is said to be *first countable* if every point of $X$ has a countable basis.
(2) $X$ is said to be *second countable* if $X$ has a countable basis.
(3) $X$ is said to be *Lindelöf* if any open cover of $X$ has a countable subcover.
(4) $X$ is *separable* if it has a countable dense subset.

**Example 1.34.** $\mathbb{R}^n$ is first countable, which is clear. In general, any metric space is first countable.

**Example 1.35.** $\mathbb{R}^n$ and $\mathbb{R}^\omega$ are second countable.

**Proposition 1.34.** *The following hold.*

(1) *A subspace of a first countable (respectively second countable) space is first countable (respectively, second countable).*

(2) *A countable product of first countable (respectively second countable) spaces if first countable (respecitively second countable).*

*Proof.* (1) is trivial. For (2), just use the fact that the set of finite subsets of a countable set is countable. ∎

**Proposition 1.35.** *Second countability implies all other countability axioms.*

*Proof.* If $X$ is second countable, then it is clearly first countable. This is trivial.

Next, we show that every second countable space is Lindelöf. So, let $\mathcal{O}$ be any open cover of $X$. Let $\{B_n\}_{n\in\mathbb{N}}$ be a basis for $X$. For each $n \geq 1$, choose $A_n$ (if possible) such that $B_n \subseteq A_n$ (if this is not possible, then $A_n$ is undefined). We claim that $\{A_n\}$ covers $X$. To show this, suppose $x \in X$. Now there is some $A \in \mathcal{O}$ such that $x \in A$. Now, there is a $B_n$ such that $x \in B_n \subseteq A$ (this is true because $\{B_i\}$ is a basis). This means that $A_n$ is defined and $x \in A_n$. So, we have extracted a countable subcover.

Next, we show that any second countable space is separable. Let $\{B_n\}$ be a basis of $X$. Choose $x_n \in B_n$ (if $B_n$ is non-empty). Then $\{x_n\}$ is a countable dense subset. This completes the proof. ∎

**Example 1.36.** We show that $\mathbb{R}_l$ satisfies all countability axioms except the second.

(1) $\mathbb{R}_l$ is first countable: let $x \in \mathbb{R}_l$ and take $\left\{\left[x, x+\frac{1}{n}\right)\right\}_{n\in\mathbb{N}}$.
(2) $\mathbb{R}_l$ is separable because $\mathbb{Q} \subseteq \mathbb{R}_l$ is a countable dense subset.
(3) $\mathbb{R}_l$ is Lindelöf: it suffices to show that an open cover of $\mathbb{R}_l$ by basic open sets has a countable subcover. Let $\{[a_\alpha, b_\alpha)\}_{\alpha\in J}$ be an open cover of $\mathbb{R}_l$. Let
$$C := \bigcup_{\alpha\in J}(a_\alpha, b_\alpha)$$
Then, we show that $\mathbb{R}\setminus C$ is countable. To show this, suppose $x \in \mathbb{R}\setminus C$. Then $x = a_\alpha$ for some $\alpha \in J$. Choose $q_x \in \mathbb{Q}$ such that $q_x \in (a_\alpha, b_\alpha)$. So we see that
$$(x, q_x) \subseteq (x, b_\alpha) = (a_\alpha, b_\alpha)$$
Let $x, y \in \mathbb{R} \setminus C$.such that $x < y$, say $y = a_\beta$ for some $\beta \in J$. We show that $q_x < q_y$. For the sake of contradiction, suppose $q_y \leq q_x$. This implies that $y < q_y \leq q_x$, implying that $y \in (x, q_x) \subseteq C$, which is a contradiction. Hence, it follows that $q_y > q_x$. So the map $\mathbb{R} - C \to \mathbb{Q}$ given by $x \mapsto q_x$ is injective, and hence $\mathbb{R} - C$ is countable.
Next, we claim that $C$ is covered by countably many $(a_\alpha, b_\alpha)$. We think of $C$ as a subspace of $\mathbb{R}$. Since $\mathbb{R}$ is second countable, so is $C$ by **Proposition** 1.34. Hence **Proposition** 1.35 implies that $C$ is Lindelöf. So, $C$ is covered by a countable subcollection of $\{(a_\alpha, b_\alpha)\}$, say
$$C = \bigcup_{n\geq 1}(a_n, b_n)$$
Then it follows that
$$\{[a_n, b_n)\}_{n\geq 1} \cup \{[a_\alpha, b_\alpha)\}_{\alpha\in\mathbb{R}\setminus C}$$
is a countable subcover of $\mathbb{R}_l$, and hence $\mathbb{R}_l$ is Lindelöf.
(4) $\mathbb{R}_l$ is not second countable: Let $\mathcal{B}$ be a basis for $\mathbb{R}_l$. For $x \in \mathbb{R}_l$, let $B_x \in \mathcal{B}$ be such that $x \in B_x \subseteq [x, x+1)$. We claim that for $x \neq y$, $B_x \neq B_y$, because we clearly see that $x = \inf B_x$ and $y = \inf B_y$. So the map $\mathbb{R}_l \to \mathcal{B}$ given by $x \to B_x$ is injective, and hence $\mathcal{B}$ is uncountable.

**Example 1.37.** In this example, we will show that $\mathbb{R}_l^2$ is not Lindelöf. Let

$$L := \{(x, -x) \mid x \in \mathbb{R}_l\} \subseteq \mathbb{R}_l^2$$

We see that $L \subseteq \mathbb{R}_l^2$ is closed, because its complement is open. Take the following open cover of $\mathbb{R}_l^2$:

$$\{\mathbb{R}_l^2 \setminus L\} \cup \{[a, a+1) \times [-a, -a+1) \mid a \in \mathbb{R}_l\}$$

This has no *countable subcover*, since for every point of $L$, we need a distinct element of the open cover and $L$ is uncountable.

**Example 1.38.** Let $X = I^2$ in the dictionary order topology. By **Theorem** 1.27 we know that $X$ is compact. Hence $X$ is Lindelöf. But we claim that $Y := I \times (0,1)$ is *not* Lindelöf. Observe that $Y$ is covered by the sets $\{\{x\} \times (0,1)\}_{x \in I}$ and there is no proper subcover, and this open cover is clearly uncountable. This example shows that not all subspaces of Lindelöf spaces are Lindelöf.

1.15. **Separation Axioms.** An example of a separation axiom is *Hausdorffness*. We will look at some more separation axioms here.

**Definition 1.24.** Let $X$ be a topological space in which all singletons are closed.

(1) $X$ is *regular* if given a point $x \in X$ and a closed set $A \subseteq X$ such that $x \notin A$, there exist disjoint open sets $U$ and $V$ such that $x \in U$, $A \subseteq V$.
(2) $X$ is said to be *normal* if given two disjoint closed sets $A$ and $B$ in $X$, there exist disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

Clearly, normal $\implies$ regular $\implies$ Hausdorff.

**Remark 1.35.1.** In the above definition, we wanted to have stronger properties than Hausdorffness, and hence we required all singletons to be closed because in a Hausdorff space all singletons are closed.

**Exercise 1.9.** Show that metric spaces are normal. In particular, $\mathbb{R}^n$ is normal.

**Solution.** This follows from part (2) of **Theorem** 1.38.

**Proposition 1.36.** *Let $X$ be a topological space where singletons are closed. Then the following are true.*

(1) *$X$ is regular if and only if given $x \in X$ and a neighborhood $U$ of $x$, there exists a neighborhood $V$ of $x$ such that $\overline{V} \subseteq U$.*
(2) *$X$ is normal if and only if given a closed set $A \subseteq X$ and an open set $U \supseteq A$, there is an open set $V$ such that $A \subseteq V \subseteq \overline{V} \subseteq U$.*

*Proof.* First we prove (1). So suppose the given property is true. Let $x \in X$ and $A \subseteq X$ be a closed set such that $x \notin A$. Now, the set $A^c$ is open, and contains $x$. So, there is some open set $V$ such that $x \in V \subseteq \overline{V} \subseteq A^c$. Let $U = (\overline{V})^c$, and hence $U$ is an open set containing $A$. Clearly, $U, V$ are the required disjoint open sets.

Conversely, suppose $X$ is regular, and let $x \in X$ and an open neighborhood $U$ of $x$ be given. This means that $U^c$ is a closed set such that $x \notin U^c$. So, there are disjoint open sets $A, B$ in $X$ such that $x \in A$ and $U^c \subseteq B$. Clearly, this means that $x \in A \subseteq \overline{A} \subseteq U$, and hence this shows that the given property is true. This completes the proof.

Next, we prove (2). Suppose $X$ is normal. Let $A$ be a closed set, and let $U$ be an open set such that $A \subseteq U$. Let $B = X - U$. So, we see that $A, B$ are disjoint. By the normality of $X$, there are open disjoint sets $V, W$ such that $A \subseteq V$ and $B \subseteq W$.

We claim that this $V$ works; clearly, $A \subseteq V$. To prove $\overline{V} \subseteq U$, we will show that $\overline{V} \cap B = \phi$. This is true because of the following: if $y \in B$, then $W$ is a neighborhood of $y$ such that $W \cap V = \phi$, which implies that $y$ is neither a point of $V$ nor a limit point of $V$, and hence $y \notin \overline{V}$.

Conversely, suppoes the given condition is true, and we will show that $X$ is normal. So, let $A, B$ be any disjoint closed sets in $X$. Let $U = X - B$. Then we see that $A \subseteq U$. By hypothesis, there is some open set $V$ such that $A \subseteq V \subseteq \overline{V} \subseteq U$. Let $W = X - \overline{V}$. Then $V \cap W = \phi$, and $A \subseteq V$, $B \subseteq W$. ■

**Proposition 1.37.** *The following hold.*

(1) *A subspace of a regular space is regular.*
(2) *Any product of regular spaces is regular.*

**Remark 1.37.1.** Recall that we proved these for Hausdorff spaces in **Proposition** 1.6 and **Exercise** 1.7.

*Proof.* Let us prove (1) first. So let $X$ be any regular space, and let $Y \subseteq X$ be any subspace. We see that singletons are closed in $Y$. Let $x \in Y$ and $B \subseteq Y$ be closed with $x \notin B$. Because $\overline{B} \cap Y = B$ (since $B$ is closed in $Y$), we see that $x \notin \overline{B}$. By the regularity of $X$, there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $\overline{B} \subseteq V$. Then just take the sets $U \cap Y$ and $V \cap Y$.

Next, we prove (2). Let $\{X_\alpha\}_{\alpha \in J}$ be a family of regular spaces. Let $X = \prod_{\alpha \in J} X_\alpha$. To prove that $X$ is regular, we will be using part (1) of **Proposition** 1.36. Let $x = (x_\alpha) \in X$ and let $U \subseteq X$ be an open neighborhood of $X$. Choose a basic open set $\prod U_\alpha$ such that $x \in \prod U_\alpha \subseteq U$. For each $\alpha$, choose a neighborhood $V_\alpha$ of $x_\alpha$ such that $x_\alpha \in V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$ (possible by **Proposition** 1.36, because each $X_\alpha$ is regular). If $U_\alpha = X_\alpha$ set $V_\alpha = X_\alpha$. Let $V = \prod V_\alpha$. Clearly, $V \subseteq X$ is open. Also note that

$$\overline{V} = \prod_{\alpha \in J} \overline{V_\alpha}$$

and hence $x \in V \subseteq \overline{V_\alpha} \subseteq V$, as required. Finally, since $X_\alpha$ is Hausdorff for all $\alpha \in J$, $X$ is also Hausdorff, which means that singletons in $X$ are closed. This shows that $X$ is regular, completing the proof. ■

**Example 1.39.** Consider the space $\mathbb{R}_K$, where the basic open sets were of the form $(a, b)$ and $(a, b) - K$ where

$$K = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$$

We claim that $\mathbb{R}_K$ is Hausdorff but not regular. It is clear that $\mathbb{R}_K$ is Hausdorff, because $\mathbb{R}$ is Hausdorff and $\mathbb{R}_K$ is finer than $\mathbb{R}$.

Next, it is easy to see that $K$ is closed in $\mathbb{R}_K$ (try to prove this if you don't see this). We show that 0 and $K$ can't be separated by disjoint open sets. Suppose there are disjoint open sets $U$ and $V$ such that $0 \in U$ and $K \subseteq V$. Choose a basic open neighborhood of 0 such that $W \subseteq U$. Then clearly, $W = (a, b) - K$ for some $a, b \in \mathbb{R}$. Choose $n \gg 0$ such that $\frac{1}{n} \in (a, b)$, and hence $\frac{1}{n} \in V$. Now, choose a basic neighborhood $(c, d)$ of $\frac{1}{n}$ such that $(c, d) \subseteq V$. Finally, choose $z$ such that $z < \frac{1}{n}$ and $z > \max\{c, \frac{1}{n+1}\}$. Then, $z \in U \cap V$, and this is a contradiction.

**Example 1.40.** In this exercise, we show that $\mathbb{R}_l$ is normal. Let $A, B$ be disjoint closed sets. For each $a \in A$, choose a basic open set $[a, x_a)$ not intersecting $B$ (possible because $B^c$ is open). Similarly, we choose some $[b, x_b)$ for all $b \in B$ such that these basic open sets do not intersect $A$. Let

$$U = \bigcup_{a \in A}[a, x_a) \quad , \quad V = \bigcup_{b \in B}[b, x_b)$$

Clearly $A \subseteq U$, $B \subseteq V$ and both $U, V$ are open. We show that $U$ and $V$ are disjoint. If they are not disjoint, then there are $a \in A$ and $b \in B$ such that $[a, x_a) \cap [b, x_b) \neq \phi$. Let $y \in [a, x_a) \cap [b, x_b)$. Clearly, $y \neq a$ and $y \neq b$. So, $a < y < x_a$ and $b < y < x_b$. Without loss of generality suppose $a < b$. Then, we see that $a < b < x_a$, i.e $b \in [a, x_a)$, which is a contradiction.

**Example 1.41.** We have seen in the above example that $\mathbb{R}_l$ is normal, and hence it is regular. By **Proposition** 1.37, it follows that $\mathbb{R}_l^2$ is also regular. We claim that $\mathbb{R}_l^2$ is not normal, and this will give us an example of the fact that product of normal spaces need not be normal.

Let

$$L := \{(x, -x) \mid x \in \mathbb{R}_l\}$$

Observe that $L \subseteq \mathbb{R}_l^2$ is a discrete set, i.e every point in $L$ is open in $L$, and that $L$ is a closed subset of $\mathbb{R}_l^2$. For the sake of contradiction, assume that $\mathbb{R}_l^2$ is normal. Now, if $A \subseteq L$, then clearly $A$ is closed in $L$ (since $L$ is a discrete set). Hence, $A$ and $L - A$ are both closed in $L$, and since $L$ is closed in $\mathbb{R}_l^2$, it follows that $A$ and $L - A$ are closed in $\mathbb{R}_l^2$. So by normality, there exist disjoint open sets $U_A$ and $V_A$ such that $A \subseteq U_A$ and $L - A \subseteq V_A$.

Now, let $D := \mathbb{Q}^2 \subseteq \mathbb{R}_l^2$. So, $D$ is dense in $\mathbb{R}_l^2$. Define a map $\theta : \mathcal{P}(L) \to \mathcal{P}(D)$ as follows: for $A \subseteq L$, we put

$$\theta(A) = \begin{cases} \phi & , A = \phi \\ D & , A = L \\ D \cap U_A & , A \neq \phi, A \neq L \end{cases}$$

We claim that $\theta$ is injective, which will be the required contradiction (since $L$ is uncountable).

To prove this, suppose $A \subseteq L$ such that $A \neq \phi$ and $A \neq L$. Then $\theta(A) = D \cap U_A$ is non-empty (because $D$ is dense in $\mathbb{R}_l^2$) and $\theta(A) \neq D$ because $D \cap V_A \neq \phi$. Therefore it remains to check that if $A, B$ are proper non-empty subsets of $L$ with $A \neq B$, then $\theta(A) \neq \theta(B)$. Without loss of generality, let $x \in A \subseteq U_A$ such that $x \notin B$. Then $x \in L - B \subseteq V_B$. Hence $x \in U_A \cap V_B$ and so $U_A \cap V_B$ is a non-empty, open subset of $\mathbb{R}_l^2$. Hence $D \cap U_A \cap V_B \neq \phi$. But then $D \cap U_A \neq D \cap U_B$: $y \in D \cap U_A \cap V_B$ implies that $y \in D \cap U_A$ and $y \notin D \cap U_B$.

Hence $\theta : \mathcal{P}(L) \to \mathcal{P}(D)$ is injective, and clearly this is a contradiction. So, $\mathbb{R}_l^2$ is not normal.

**Theorem 1.38.** *The following are true.*

1. *A regular space with a countable basis is normal.*
2. *A metrizable space is normal.*
3. *A compact Hausdorff space is normal.*
4. *A well-ordered set (in order topology) is normal. Infact, any order topology is normal.*

*Proof.* We will prove each of these statemenets one by one.

(1) Let us prove (1) first. Let $X$ be a regular space with a countable basis $\mathcal{B}$. Let $A, B$ be disjoint closed sets in $X$. For $x \in A$, there exists a neighborhood $V$ of $x$ such that $V \cap B = \phi$ (true because $B$ is closed). Now, by **Proposition** 1.36 part (1), there is an open neighborhood $U$ of $x$ such that $x \in U \subseteq \overline{U} \subseteq V$. So, there exists $U_n \in \mathcal{B}$ such that $x \in U_n \subseteq U$. In other words, we can cover $A$ by $\{U_n\}_{n\geq 1}$ such that

$$\overline{U_n} \cap B = \phi \qquad \forall n \in \mathbb{N}$$

Similarly, we cover $B$ by $\{V_n\}_{n\geq 1}$ such that $\overline{V_n} \cap A = \phi$ for all $n \in \mathbb{N}$. It may happen that

$$\left(\bigcup_n U_n\right) \cap \left(\bigcup_n V_n\right) \neq \phi$$

For each $n$ put

$$U_n' := U_n - \bigcup_{i=1}^{n} \overline{V_i}$$
$$V_n' := V_n - \bigcup_{i=1}^{n} \overline{U_i}$$

It is clear that each $U_n'$ and $V_n'$ is an open set. We claim that $\bigcup_n U_n'$ and $\bigcup_n V_n'$ do the job for us, i.e they are disjoint open sets covering $A$ and $B$ respectively. Now, observe that

$$\begin{aligned} x \in A &\implies x \in U_n \text{ for some } n, \text{ but } x \notin \overline{V_i} \text{ for all i} \\ &\implies x \in U_n' \end{aligned}$$

and hence $A \subseteq \bigcup_n U_n'$. Similarly, $B \subseteq \bigcup_n V_n'$. So, these sets cover $A$ and $B$ respectively.

Next, suppose $x \in (\bigcup_n U_n') \cap (\bigcup_n V_n')$. Then $x \in U_j' \cap V_k'$ for some $j, k$. Suppose $j \leq k$. Then,

$$\begin{aligned} x \in U_j' &\implies x \in U_j \\ x \in V_k' &\implies x \notin \overline{U_j} \end{aligned}$$

and this is a contradiction. This completes the proof of (1).

(2) Next, let us complete the proof of (2). Let $X$ be a metrizable space with metric $d$. Let $A, B$ be disjoint closed sets in $X$. For each $a \in A$, choose $\epsilon_a > 0$ such that $B(a, \epsilon_a) \cap B = \phi$. Similarly, for all $b \in B$, choose $\epsilon_b > 0$ such that $B(b, \epsilon_b) \cap A = \phi$. Let

$$U := \bigcup_{a \in A} B(a, \epsilon_a/2) \quad , \quad V := \bigcup_{b \in B} B(b, \epsilon_b/2)$$

Then we see that $A \subseteq U$ and $B \subseteq V$, and that $U, V \subseteq X$ are both open. Next, we show that $U, V$ are disjoint. Observe that

$$\begin{aligned} x \in U \cap V &\implies x \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2) \text{ for some } a \in A, b \in B \\ &\implies d(a,b) \leq d(a,x) + d(b,x) < \frac{\epsilon_a + \epsilon_b}{2} \end{aligned}$$

Without loss of generality, say $\epsilon_a \leq \epsilon_b$. Then $d(a,b) < \epsilon_b$. But then $a \in B(b, \epsilon_b)$, which is not possible.

(3) Next, we will prove (3). So, let $X$ be a compact Hausdorff space. First, we will show that $X$ is regular. Let $x \in X$ be a point and let $A \subseteq X$ be a closed set such that $x \notin A$. Since $X$ is compact and $A$ is closed in $X$, we see that $A$ is also compact. Now, let $y \in A$. So $x \neq y$, and hence there are disjoint open sets $U_y$ and $V_y$ such that $x \in U_y$ and $y \in V_y$. The collection $\{V_y\}_{y \in A}$ is an open cover of $A$, and hence there is a finite subcover, say

$$A \subseteq V_{y_1} \cup \ldots \cup V_{y_n}$$

Now, consider the sets $U_{y_1} \cap \ldots \cap U_{y_n}$ and $V_{y_1} \cup \ldots \cup V_{y_n}$. Clearly these two sets are disjoint open sets, and $x \in U_{y_1} \cap \ldots \cap U_{y_n}$. So, it follows that $X$ is a regular space.

Now, we come to the main proof. Let $A, B$ be disjoint closed sets in $X$. For $a \in A$, let $U_a$, $V_a$ be disjoint open sets such that $a \in U_a$ and $B \subseteq V_a$ (we get this using regularity). Now, $A$ is compact, being a closed subset of the compact set $X$. So, $A \subseteq U_{a_1} \cup \ldots \cup U_{a_n}$ for some $a_1, \ldots, a_n \in A$. Now, put

$$U := U_{a_1} \cup \ldots \cup U_{a_n}$$
$$V := V_{a_1} \cap \ldots \cap V_{a_n}$$

So, $U, V$ separate $A, B$, and hence $X$ is normal.

(4) Skip this proof.

■

**Example 1.42.** If $J$ is uncountable, then $\mathbb{R}^J$ is not normal (see Munkres for a proof). This shows that product of normal spaces need not be normal. But, we will see that $[0,1]^J$ is compact (Tychonoff Theorem). So, $[0,1]^J$ is normal, but one of it's subspaces is not.

1.16. **Urysohn's Lemma.** In this section, we will see a very useful separation result.

**Theorem 1.39** (**Urysohn's Lemma**). *Let $X$ be a normal space, and let $A, B$ be disjoint closed subsets of $X$. Let $a < b \in \mathbb{R}$. Then there exists a continuous function $f : X \to [a, b]$ such that*

$$f(A) = \{a\} \quad , \quad f(B) = \{b\}$$

**Remark 1.39.1.** This is also called *separating $A, B$ by a continuous function*.

*Proof.* Without loss of generality, we may assume that $a = 0$ and $b = 1$. We will prove the theorem in several steps.

**Step 1:** We will construct open sets in $X$ indexed by $\mathbb{Q}$. Let $P = [0,1] \cap \mathbb{Q}$. The goal is to define open sets $U_p \subseteq X$ for all $p \in P$ such that

$$(*) \qquad p < q \implies \overline{U_p} \subseteq U_q$$

First, arrange the elements of $P$ in an infinite sequence starting with $1, 0$, i.e the first two elements of the sequence must be 1 and 0 respectively. Let

$$U_1 = X - B$$

We see that $A \subseteq U_1$. Since $X$ is normal, there is an open set $U_0$ such that $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$. For the rationals $0, 1$, statement $(*)$ is clearly true.

Suppose we have constructed the first $n$ open sets, where $n \geq 2$, such that these open sets satisfy the condition in $(*)$. Let $r$ be the next element of $P$. Then $r \neq 0, 1$. Let $P_{n+1}$ be the first $n+1$ elements of $P$. Order the elements of $P_{n+1}$ by the usual order in $\mathbb{Q}$. In this order, let $p$ be the immediate predecessor of $r$, and let $q$ be the immediate successor of $r$. We have already defined $U_p, U_q$ such that $\overline{U_p} \subseteq U_q$. By normality, there is an open set $U_r$ such that

$$\overline{U_p} \subseteq U_r \subseteq \overline{U_r} \subseteq U_q$$

Now, let $s \in P_{n+1}$.

(1) Suppose $s < r$. We need to check that $\overline{U_s} \subseteq U_r$. We know that $s \leq p$ and hence $\overline{U_s} \subseteq U_p \subseteq U_r$.
(2) If $s > r$, then we know that $s \geq q$. So $\overline{U_q} \subseteq U_s$ and hence $\overline{U_r} \subseteq \overline{U_q} \subseteq U_s$.

**Step 2:** Define

$$U_p := \phi \quad \forall p \in \mathbb{Q}, \quad p < 0$$
$$U_p := X \quad \forall p \in \mathbb{Q}, \quad p > 1$$

So we have $\{U_p \subseteq X \text{ open} \mid p \in \mathbb{Q}\}$ satisfying $(*)$.

**Step 3:** For $x \in X$, let

$$Q(x) := \{p \in \mathbb{Q} \mid x \in U_p\} \subseteq [0, \infty)$$

So each $Q(x)$ is bounded below, and clearly each $Q(x) \neq \phi$. Define

$$f(x) := \inf Q(x)$$

Clearly, $f(x) \in [0, 1]$ for all $x \in X$.

**Step 4:** We claim that $f$ is the function we are looking for. Observe that

$$\begin{aligned} x \in A &\implies 0 \in Q(x) \\ &\implies f(x) = \inf Q(x) = 0 \\ &\implies f(A) = \{0\} \end{aligned}$$

Similarly, we have that

$$x \in B \implies x \notin U_p \ \forall p \leq 1 \text{ (since } U_1 = X - B)$$

Also, we see that $x \in B \implies x \in U_p$ for all $p > 1$. So this means that

$$\begin{aligned} &f(x) = \inf Q(x) = 1 \\ \implies &f(B) = \{1\} \end{aligned}$$

Finally, we show that $f$ is continuous, and that will complete the proof of the claim. We will show the following: if $r \in [0, 1] \cap \mathbb{Q}$, then

(1) $x \in \overline{U_r} \implies f(x) \leq r$.
(2) $x \notin U_r \implies f(x) \geq r$.

This is easy to prove: if $x \in \overline{U_r}$ then $x \in U_s$ for all $s \geq r$ and hence $f(x) \leq r$. If $x \notin U_r$, then $x \notin U_s$ for all $s < r$ and hence $f(x) \geq r$.

Now, let $x_0 \in X$. Let $(c, d) \in \mathbb{R}$ be an open interval containing $f(x_0)$. We will find a neighborhood of $U$ of $x_0$ such that $f(U) \subseteq (c, d)$ (which will show that $f^{-1}(c, d)$ is open, and hence that $f$ is continuous). Choose rational numbers $p, q$ such that $c < p < f(x_0) < q < d$. We claim that $U := U_q \setminus \overline{U_p}$ works. So first, we show that $x_0 \in U$. Observe that $f(x_0) < q$, and hence by point (2) above, it must be true that $x \in U_q$. Similarly, we know that $f(x_0) > p$, and hence by point (1) above, it must be true that $x \notin \overline{U_p}$. This implies that $x_0 \in U_q \setminus \overline{U_p} = U$, and hence $U$ is a neighborhood of $x_0$. Next, we show that $f(U) \subseteq (c, d)$. Suppose $x \in U$. Then the following hold.

(1) $x \in U_q$, and hence $x \in \overline{U_q}$, which means $f(x) \leq q$ by point number (1) above.
(2) $x \notin \overline{U_p}$, which implies that $x \notin U_p$ and hence $f(x) \geq p$ by point (2) above.

Hence $p \leq f(x) \leq q$, which implies that $f(x) \in (c, d)$. This completes the proof. ■

**Definition 1.25.** Let $X$ be a topological space in which every singleton set is closed. $X$ is said to be *completely regular* if given a point $x \in X$ and a closed set $A \subseteq X$ such that $x \notin A$, there is a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

**Definition 1.26.** A space $X$ is said to be *completely normal* if every subspace of $X$ is normal.

We have

$T_5$(completely normal) $\iff$ $T_4$(normal) $\implies$ $T_3$(regular) $\implies$ $T_2$(Hausdorff) $\implies$ $T_1$

The fact that $T_5 \implies T_4$ is easy. On the other hand, the fact that $T_4 \implies T_5$ is hard to show, but we have shown this in **Urysohn's Lemma** 1.39.

**Exercise 1.10.** Find counterexamples to prove that $T_1 \not\Rightarrow T_2$, $T_2 \not\Rightarrow T_3$, $T_3 \not\Rightarrow T_4$ and $T_{3.5} \not\Rightarrow T_4$. Is $T_3 \implies T_{3.5}$ true?

**Solution.** To be completed. Counterexample for $T_3 \not\Rightarrow T_4$ is $\mathbb{R}_l^2$.

**Exercise 1.11.** Is it true that Hausdorffness implies complete Hausdorfness?

**Solution.** To be completed.

1.17. **Urysohn Metrization Theorem.** Here is another important result.

**Theorem 1.40** (**Urysohn Metrization Theorem**). *Every regular space with a countable basis is metrizable.*

**Remark 1.40.1.** Regularity is a necessary condition for metrizability (because every metrizable space is normal by **Theorem** 1.38, and hence regular), but having a countable basis is not.

*Proof.* Let $X$ be a regular space with a countable basis. We will prove that $X$ is a subspace of a metric space. In fact, we will show that $X$ is homeomorphic to a subspace of $\mathbb{R}^\omega$. We will be using the fact that $\mathbb{R}^\omega$ is metrizable. A proof of this can be found in Munkres.

**Step 1:** *There exists a countable collection of continuous functions $f_n : X \to [0, 1]$ such that given any $x_0 \in X$ and any neighborhood $U$ of $x_0$, there exists $n$ such that $f_n(x_0) > 0$ and $f_n(X \setminus U) = \{0\}$.*

Let us prove **Step 1**. By **Theorem** 1.38 part (1), we know that $X$ is a normal space. Let $\{B_n\}$ be a countable basis of $X$. For each pair $n, m$ such that $\overline{B_n} \subseteq B_m$, apply **Uryosohn's Lemma** 1.39 to $\overline{B_n}$ and $X - B_m$ to obtain a continuous function $g_{n,m} : X \to [0, 1]$ such that $g_{n,m}(\overline{B_n}) = 1$ and $g_{n,m}(X - B_m) = 0$. We claim that $\{g_{n,m}\}$ is the desired collection. Note that this collection is countable. Let $x_0 \in X$ and let $U \subseteq X$ be a neighborhood of $x_0$. Since $\{B_n\}$ is a basis, there is some $B_m$ such that $x_0 \in B_m \subseteq U$. Now, regularity implies that there is some $B_n$ such that $x_0 \in B_n \subseteq \overline{B_n} \subseteq B_m \subseteq U$. So, $g_{n,m}$ is defined for this pair $n, m$. Now $g_{n,m}(x_0) = 1$

and $g_{n,m}(X - B_m) = \{0\}$ which implies that $g_{n,m}(X - U) = \{0\}$. This completes the proof of **Step 1**.

**Step 2:** *Consider the function $F : X \to \mathbb{R}^\omega$ given by $x \mapsto (f_1(x), f_2(x), \cdots)$. Then $F$ is an embedding of $X$ into $\mathbb{R}^\omega$.*

To prove **Step 2**, we make the following observations.

(1) $F$ is continuous, since each $f_n$ is.

(2) $F$ is injective: let $x \neq y \in X$. By regularity, there is an open set $U \subseteq X$ such that $x \in U$ and $y \notin U$. By **Step 1**, there is some $n \in \mathbb{N}$ such that $f_n(x) = 1$ and $f_n(y) = 0$. So $F(x) \neq F(y)$.

(3) $F : X \to F(X)$ is an open map: Let $Z := F(X)$ and let $U \subseteq X$ be open. We will prove that $F(U)$ is open in $\mathbb{R}^\omega$. Let $z_0 \in F(U)$. Then there is some $x_0 \in U$ such that $F(x_0) = z_0$.

Choose $N$ such that $f_N(x_0) > 0$ and $f_N(X - U) = \{0\}$, which is possible by **Step 1**. Let

$$V = \mathbb{R} \times \mathbb{R} \times \cdots \times (0, \infty) \times \mathbb{R} \times \cdots$$

where the factor $(0, \infty)$ occurs at the $N$th coordinate. Clearly, $V$ is an open set in $\mathbb{R}^\omega$. Let $W = V \cap Z$, and hence $W$ is open in $Z$. We claim that $z_0 \in W \subseteq F(U)$, and this will prove that $F(U)$ is open. First, let us show that $z_0 \in W$. Let $\pi_N : \mathbb{R}^\omega \to \mathbb{R}$ be the projection map. Then

$$\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0$$

and this implies that $z_0 \in V$, and hence $z_0 \in W$. Now, we show that $W \subseteq F(U)$. Observe that

$$\begin{aligned} z \in W &\implies z = F(x) \qquad \text{for some } x \in X \text{ and } \pi_N(z) > 0 \\ &\implies \pi_N(z) = f_N(x) > 0 \\ &\implies x \in U \qquad \text{(because } f_N(X - U) = \{0\}) \\ &\implies z = F(x) \in F(U) \\ &\implies W \subseteq F(U) \end{aligned}$$

Hence, $F : X \to F(X)$ is an open map.

The three points above show that $F : X \to \mathbb{R}^\omega$ is an embedding of $X$ into $\mathbb{R}^\omega$, and hence $X$ is metrizable. This completes the proof. ■

**Remark 1.40.2.** In the above proof, we constructed an embedding $F : X \to \mathbb{R}^\omega$. Infact, the *same* proof as above gives the following general result.

**Theorem 1.41.** *Let $X$ be a space in which singletons are closed. Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of continuous functions $f_\alpha : X \to \mathbb{R}$ satisfying the following: given $x_0 \in X$ and a neighborhood $U \subseteq X$ of $x_0$, there is some $\alpha \in J$ such that $f_\alpha(x_0) > 0$ and $f_\alpha(X - U) = \{0\}$. Then, the function $F : X \to \mathbb{R}^J$ defined by*

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

*is an embedding. If each $f_\alpha$ maps $X$ into $[0, 1]$, then $F$ embeds $X$ in $[0, 1]^J$.*

*Proof.* The same proof as in **Urysohn Metrization Theorem** 1.40 works. ■

**Theorem 1.42.** *A space $X$ is completely regular if and only if it is homeomorphic to a subspace of $[0, 1]^J$ for some $J$.*

*Proof.* To be completed. For the forward direction, use **Theorem** 1.41. For the backward direction, try to prove it directly. ■

1.18. **Tietze Extension Theorem.** This is one of the more important theorems about extensions of continuous functions.

**Theorem 1.43** (**Tietze Extension Theorem**). *Let $X$ be a normal space and let $A \subseteq X$ be closed. Let $a, b \in \mathbb{R}$.*

1. *Any continuous map $A \to [a, b]$ can be extended to a continuous map $X \to [a, b]$.*
2. *Any continuous map $A \to \mathbb{R}$ can be extended to a continuous map $X \to \mathbb{R}$.*

*Proof.* We will prove this theorem in steps.

**Step 1:** *Let $f : A \to [-r, r]$ be a continuous function, where $r > 0$ is any real number. Then there exists a continuous function $g : X \to \mathbb{R}$ such that*

1. *$|g(x)| \leq r/3$ for all $x \in X$ and*
2. *$|g(a) - f(a)| \leq 2r/3$ for all $a \in A$.*

Let's prove **Step 1**. Let $I_1 = [-r, -r/3]$, $I_2 = [-r/3, r/3]$ and $I_3 = [r/3, r]$. Let $B = f^{-1}(I_1)$ and $C = f^{-1}(I_3)$. Since $f$ is continuous, $B, C$ are disjoint closed subsets of $A$, and hence of $X$ (because $A$ is closed). Now we apply **Urysohn's Lemma** 1.39 to the sets $B, C$: there exists a continuous function $g : X \to [-r/3, r/3]$ such that $g(B) = \{-r/3\}$ and $g(C) = \{r/3\}$. We claim that $g$ satisfies the required properties (1) and (2). It is clear that $g$ satisfies (1). Next, let $a \in A$. There are three possibilities:

- The first possibility is $a \in B$. In this case, we see that $g(a) = -r/3$ and $f(a) \in I_1$, which implies that
$$|g(a) - f(a)| \leq 2/3r$$
- The second possibility is $a \in C$. The proof here is the same as that in the case $a \in B$.
- The third and final possibility is $a \notin B, a \notin C$. In this case, we see that $f(a) \in I_2$ and $g(a) \in I_2$. Again, we have
$$|g(a) - f(a)| \leq 2/3r$$
and hence we are done.

This completes the proof of **Step 1**.

Next, let us prove part (1) of the theorem. Without loss of generality suppose $a = -1$ and $b = 1$. We apply **Step 1** with $r = 1$ to obtain a continuous function $g_1 : X \to [-1/3, 1/3]$ which satisfies properties (1) and (2) of **Step 1**. Clearly, we see that $f - g_1 : A \to [-2/3, 2/3]$ is a continuous function. Again, we apply **Step 1** to $f - g_1$ to get a continuous function $g_2 : X \to [-2/9, 2/9]$ which satisfies properties (1) and (2) of **Step 1**, i.e

$$|g_2(x)| \leq \frac{2}{9} \qquad \forall x \in X \text{ and}$$
$$|f(a) - g_1(a) - g_2(a)| \leq \left(\frac{2}{3}\right)^2 \qquad \forall a \in A$$

Continuing this way, we can obtain functions $g_n : X \to \mathbb{R}$ such that

$$|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} \qquad \forall x \in X$$

$$|f(a) - g_1(a) - ... - g_n(a)| \leq \left(\frac{2}{3}\right)^{n} \qquad \forall a \in A$$

Now define the function

$$g(x) = \sum_{n=1}^{\infty} g_n(x) \qquad \forall x \in X$$

It is easy to see that the series on the RHS is convergent, as it is dominated by a geometric sum. Infact, by the Weierstrass $M$-test, we see that this series converges uniformly, and hence $g$ is a continuous function. Now, we claim that $g : X \to [-1, 1]$. This is true because

$$\frac{1}{3}\sum_{n\geq 0}\left(\frac{2}{3}\right)^{n} = 1$$

and that $|g|$ is dominated by this series. So, $g : X \to [-1, 1]$ is a continuous function. Finally, we show that $g$ is an extension of $f$. For any $a \in A$ let

$$s_n(a) := \sum_{i=1}^{n} g_n(x)$$

Then, for $a \in A$ we know that $|f(a)-s_n(a)| \leq (2/3)^n$ for all $n$ and hence $s_n(a) \to f(a)$ as $n \to \infty$, which implies that

$$g(a) = f(a) \qquad \forall a \in A$$

This completes the proof of part (1) of the theorem.

Finally, we prove part (2) of the theorem. Given a continuous function $A \to \mathbb{R}$, by composing with a homeomorphism $\mathbb{R} \cong (-1, 1)$ we get a function $f : A \to (-1, 1) \subsetneq [-1, 1]$. By part (1) of the theorem, there is a continuous function $g : X \to [-1, 1]$ such that $g(a) = f(a)$ for all $a \in A$. Let $D := g^{-1}(\{-1\}) \cup g^{-1}(\{1\}) \subseteq X$. If $D = \phi$, then $g(X) \subseteq (-1, 1)$ and we are done, i.e we can get an extension of our original function by again composing with a homeomorphism. So, suppose $D \neq \phi$. $D$ is closed in $X$ and $D \cap A = \phi$. So, we can apply **Urysohn's Lemma** 1.39 to $A, D$: we get a continuous function $\phi : X \to [0, 1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$. Now define $h : X \to [-1, 1]$ by $h(x) = \phi(x)g(x)$. We show the following two things.

- $h(X) \subseteq (-1, 1)$: clearly, if $x \in D$ then $h(x) = \phi(x)g(x) = 0$. If $x \notin D$, then $|g(x)| < 1$ and hence $|h(x)| = |\phi(x)g(x)| < 1$ which implies that $h(x) \in (-1, 1)$.
- $h$ extends the function $f$: if $a \in A$, then $h(a) = \phi(a)g(a) = 1 \cdot f(a) = f(a)$.

So, $h : X \to (-1, 1)$ is an extension of $f$, and again by composing with a homeomorphism, we can obtain an extension of our oringinal function. This completes the proof. ■

**Exercise 1.12.** Find a space $X$ and a closed subset $A \subseteq X$ for which the extension theorem does not hold.

**Solution.** To be completed.

**Exercise 1.13.** Show that the **Tietze Extension Theorem** 1.43 implies **Urysohn's Lemma** 1.39.

**Exercise 1.14.** Let $X = \mathbb{R}$, $A = (0, 1)$ and $B = (1, 2)$. Show that $A, B$ can't be separated by a continuous function. Show that the **Tietze Extension Theorem** 1.43 fails for $A \subseteq X$.

**Exercise 1.15.** Let $X$ be a regular, second countable space. Let $U \subseteq X$ be open.

(1) Show that $U$ is a countable union of closed subsets of $X$.
(2) Show that there exists a continuous function $f : X \to [0, 1]$ such that $f(x) > 0$ for all $x \in U$ and $f(X \setminus U) = \{0\}$.

1.19. **Manifolds.** This will be a short discussion on *manifolds*.

**Definition 1.27.** Let $m > 0$ be an integer. An *$m$-manifold* is a Hausdorff, second countable space $X$ such that every point $x \in X$ has a neighborhood which is homeomorphic to an open subset of $\mathbb{R}^m$.

**Example 1.43.** The sphere $S^n \subseteq \mathbb{R}^{n+1}$ is an $n$-manifold, though this is not completely trivial. The case $n = 1$ is not hard to see.

**Theorem 1.44.** *Any connected, compact 1-manifold is homeomorphic to $S^1$. Any connected, non-compact 1-manifold is homeomorphic to $\mathbb{R}$.*

*Proof.* We won't prove these here. ∎

**Definition 1.28.** If $\phi : X \to \mathbb{R}$ is a function on a topological space $X$, the support of $\phi$, represented by $\text{supp}(\phi)$, is defined as

$$\text{supp}(\phi) = \overline{\phi^{-1}(\mathbb{R} - \{0\})}$$

So if $x \notin \text{supp}(\phi)$, then there is some neighborhood $U$ of $x$ such that $\phi(U) = \{0\}$.

**Definition 1.29.** Let $\{U_1, ..., U_n\}$ be an indexed open cover of a space $X$. A *partition of unity dominated by* $\{U_i\}$ is an indexed family of continuous functions

$$\phi_i : X \to [0, 1], \quad 1 \leq i \leq n$$

such that

(1) $\text{supp}(\phi_i) \subseteq U_i$.
(2) $\sum_{i=1}^{n} \phi_i(x) = 1$ for all $x \in X$.

**Theorem 1.45.** *Let $X$ be normal, and let $\{U_1, ..., U_n\}$ be an open cover of $X$. Then there exists a partition of unity dominated by $\{U_i\}$.*

*Proof.* We will prove this theorem in a couple of steps.

**Step 1:** *There is an open cover $\{V_1, ..., V_n\}$ of $X$ such that $\overline{V_i} \subseteq U_i$ for each $1 \leq i \leq n$.*

To show this, let $A := X - (U_1, ..., U_n)$; $A \subseteq X$ is closed, and since $\{U_i\}$ cover $X$ we see that $A \subseteq U_1$. Since $X$ is normal, there is an open set $V_1$ such that $A \subseteq V_1 \subseteq \overline{V_1} \subseteq U_1$. Since $A \subseteq V_1$ and $\{A, U_2, ..., U_n\}$ cover $X$, it follows that $\{V_1, U_2, ..., U_n\}$ cover $X$. Now, we can repeat this procedure to obtain the sets $V_i$ for each $1 \leq i \leq n$, and this completes the proof of **Step 1**.

**Step 2:** Here we will prove the theorem. So given $\{U_1, ..., U_n\}$, let $\{V_1, ..., V_n\}$ be as constructed in **Step 1**. Again by **Step 1**, choose an open cover $\{W_1, ..., W_n\}$ of $X$ such that $\overline{W_i} \subseteq V_i$ for $1 \leq i \leq n$. Now, applying **Urysohn's Lemma** 1.39 to the closed sets $\overline{W_i}$ and $X - V_i$, we get continuous functions $\psi_i : X \to [0, 1]$ for each $1 \leq i \leq n$ such that

$$\psi_i(\overline{W_i}) = \{1\} \text{ and } \psi_i(X - V_i) = \{0\}$$

This means that $\text{supp}(\psi_i) \subseteq \overline{V_i} \subseteq U_i$.

Now, for every $x \in X$, $x \in W_i$ for some $i$, and hence $\psi_i(x) = 1$ for some $i$. This means that

$$\sum_{i=1}^{n} \psi_i(x) > 0$$

for all $x \in X$. For each $1 \leq i \leq n$, define

$$\varphi_i(x) = \frac{\psi_i(x)}{\sum_{j=1}^{n} \psi_j(x)} \qquad \forall x \in X$$

We see that each $\varphi_i$ is a continuous function $X \to [0,1]$, and

(1) $\text{supp}(\varphi_i) = \text{supp}(\psi_i) \subseteq U_i$ for each $1 \leq i \leq n$.
(2) $\sum_i \varphi_i(x) = 1$ for all $x \in X$.

■

**Theorem 1.46.** *If $X$ is a compact $m$-manifold, then $X$ can be embedded in $\mathbb{R}^N$ for some $N > 0$.*

*Proof.* Let $\{U_1, ..., U_n\}$ be an open cover of $X$ such that for each $i$, we have a homeomorphism $g_i : U_i \to \mathbb{R}^m$ (such a finite cover exists because $X$ is assumed to be compact). By **Theorem** 1.38, we know that $X$ is a normal space, since it is Hausdorff and compact. By **Theorem** 1.45, there is a partition of unity $\{\phi_1, ..., \phi_n\}$ dominated by the cover $\{U_1, ..., U_n\}$. Let $A_i = \text{supp}(\phi_i)$ for each $i$. Define maps $h_i : X \to \mathbb{R}^m$ by

$$h(x) = \begin{cases} \phi_i(x) g_i(x) & x \in U_i \\ 0 & x \in X - A_i \end{cases}$$

So, each $h_i$ is a continuous function. Let $F : X \to \mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m = \mathbb{R}^{n+mn}$ be the map defined by

$$x \mapsto (\phi_1(x), ..., \phi_n(x), h_1(x), ...., h_n(x))$$

Let $N = n + mn$. We claim that $F$ is an embedding. Clearly, $F$ is continuous, because each of its component functions are continuous. It suffices to show that $F$ is injective, because $X$ is compact and $\mathbb{R}^N$ is Hausdorff (which will imply that the inverse is also continuous). Let $F(x) = F(y)$ for some $x, y \in X$. This means that

$$\phi_i(x) = \phi_i(y) \quad , \quad h_i(x) = h_i(y)$$

for each $1 \leq i \leq n$. Now, there is some $i$ such that $\phi_i(x) > 0$. Hence, $\phi_i(y) > 0$. Because $h_i(x) = h_i(y)$, this implies that $g_i(x) = g_i(y)$, which implies that $x = y$, because $g_i$ is a homeomorphism. ■

**Remark 1.46.1.** The above theorem actually holds for any $m$-manifold.

1.20. **Tychonoff Theorem.** Now we will prove one of the most important and difficult theorems in topology.

**Lemma 1.47.** *Let $X$ be a set, and let $\mathscr{A}$ be a collection of subsets of $X$ satisfying the finite intersection property. Then there is a collection $\mathscr{D}$ of subsets of $X$ such that $\mathscr{A} \subseteq \mathscr{D}$, $\mathscr{D}$ has the finite intersection property and no collection of subsets of $X$ that properly contains $\mathscr{D}$ has the finite intersection property.*

*Proof.* We will use Zorn's Lemma to prove this. We are given a collection $\mathscr{A}$ of subsets of $X$ having the finite intersection property. Let $\mathcal{A}$ be the superset consisting of all collections $\mathscr{B}$ of subsets of $X$ such that

(1) $\mathscr{A} \subseteq \mathscr{B}$ and
(2) $\mathscr{B}$ has the finite intersection property.

Note that $\mathcal{A} \neq \phi$. The order on the set $\mathcal{A}$ is given by set inclusion. Our goal is to show that $\mathcal{A}$ has a maximal element for this order, and to use Zorn's lemma, we will have to show that every chain in $\mathcal{A}$ has an upper bound. So let $\mathcal{B}$ be any chain in $\mathcal{A}$, and let

$$\mathscr{C} := \bigcup_{\mathscr{B} \in \mathcal{B}} \mathscr{B}$$

Clearly, $\mathscr{C}$ is a collection of subsets of $X$. We claim that $\mathscr{C}$ is an upper bound for $\mathcal{B}$.

- $\mathscr{A} \subseteq \mathscr{C}$ because each $\mathscr{B} \supseteq \mathscr{A}$.
- $\mathscr{C}$ has the finite intersection property: Let $C_1, ..., C_n \in \mathscr{C}$. Then $C_i \in \mathscr{B}_i$ for each $i$, where $\mathscr{B}_i \in \mathcal{B}$. Since $\mathcal{B}$ is a chain, $\{\mathscr{B}_1, ..., \mathscr{B}_n\}$ has a largest element, say $\mathscr{B}_1$. So, $C_i \in \mathscr{B}_1$ for each $i$. Since $\mathscr{B}_1$ has the finite intersection property, we see that
$$C_1 \cap ... \cap C_n \neq \phi$$
as required. Hence $\mathscr{C} \in \mathcal{A}$. Clearly, $\mathscr{C}$ is an upper bound for $\mathcal{B}$. So by Zorn's lemma, $\mathcal{A}$ has a maximal element, and this completes the proof of the lemma. ■

**Lemma 1.48.** *Let $X$ be a set, and let $\mathscr{D}$ be a collection of subsets of $X$ that is maximal with respect to the finite intersection property. Then the following are true.*

(1) *Any finite intersection of elements of $\mathscr{D}$ is an element of $\mathscr{D}$.*
(2) *If $A$ is a subset of $X$ such that $A \cap D \neq \phi$ for every $D \in \mathscr{D}$, then $A$ is an element of $\mathscr{D}$.*

*Proof.* First, let us prove part (1). Let $D_1, ..., D_n \in \mathscr{D}$ and let $B := D_1 \cap ... \cap D_n$. Consider $\mathscr{E} = \mathscr{D} \cup \{B\}$. If $\mathscr{E} = \mathscr{D}$, then we are done. So, suppose $\mathscr{E} \neq \mathscr{D}$, and hence $\mathscr{E}$ doesn't have the finite intersection property. But this is a contradiction: Let $E_1, ..., E_m \in \mathscr{E}$. Two cases are possible.

(1) $E_1, ..., E_m \in \mathscr{D}$: in this case, we have $E_1 \cap ... \cap E_n \neq \phi$, because $D$ has the finite intersection property.
(2) In the second case, suppose $E_1 = B$ without loss of generality. Then
$$E_1 \cap ... \cap E_n = D_1 \cap ... \cap D_n \cap E_2 \cap ... \cap E_m \neq \phi$$
which is true because $D$ has the finite intersection property.

Now, let us prove (2). Let $\mathscr{E} = \mathscr{D} \cup \{A\}$. Then we claim that $\mathscr{E}$ has the finite intersection property. To show this, if $D_1, ..., D_n \in \mathscr{D}$ then

$$D_1 \cap ... \cap D_n \cap A \neq \phi$$

by hypothesis. Since $\mathscr{D}$ is maximal with respect to the finite intersection property, it follows that $\mathscr{E} = \mathscr{D}$, and hence $A \in \mathscr{D}$. This completes the proof of the lemma. ■

**Theorem 1.49** (**Tychonoff Theorem**). *An arbitrary product of compact spaces is compact.*

*Proof.* Let $X_\alpha$ be a compact space for every $\alpha \in J$, where $J$ is some indexing set. Let

$$X := \prod_{\alpha \in J} X_\alpha$$

To prove that $X$ is compact, we will use the characterisation of compactness using the finite intersection property, i.e we will use the result of **Theorem** 1.26. We will show the following: if $\mathscr{A}$ is any collection of subsets of $X$ having the finite intersection property, then

$$(\star) \qquad \bigcap_{A\in\mathscr{A}} \overline{A} \neq \phi$$

and this will show that $X$ is compact, and that will complete the proof.

So, let $\mathscr{A}$ be any such collection. By Lemma 1.47, there is a collection $\mathscr{D}$ of subsets of $X$ such that $\mathscr{A} \subseteq \mathscr{D}$, $\mathscr{D}$ has the finite intersection property and $\mathscr{D}$ is maximal with respect to the finite intersection property. Now, we will show that

$$\bigcap_{D\in\mathscr{D}} \overline{D} \neq \phi$$

and that will automatically show $(\star)$.

For $\alpha \in J$, let $\pi_\alpha : X \to X_\alpha$ be the projection map. Because $\mathscr{D}$ has the finite intersection property, it follows that $\{\pi_\alpha(D) \mid D \in \mathscr{D}\}$ is a collection of subsets of $X_\alpha$ satisfying the finite intersection property. So, it follows that $\{\overline{\pi_\alpha(D)} \mid D \in \mathscr{D}\}$ is a collection of closed subsets of $X_\alpha$ having the finite intersection property. Since $X_\alpha$ is compact, there is some $x_\alpha \in X_\alpha$ such that $x_\alpha \in \overline{\pi_\alpha(D)}$ for all $D \in \mathscr{D}$. We claim that

$$x = (x_\alpha)_{\alpha\in J} \in \overline{D} \qquad \forall D \in \mathscr{D}$$

Now, fix $\beta \in J$. Let $U_\beta \in X_\beta$ be any open set. Then, $\pi_\beta^{-1}(U_\beta) \subseteq X$ is an open set. Suppose $x \in \pi_\beta^{-1}(U_\beta)$, which implies that $x_\beta \in U_\beta$. Since $x_\beta \in \overline{\pi_\beta(D)}$ for all $D \in \mathscr{D}$, it follows that $\pi_\beta(D) \cap U_\beta \neq \phi$. This means that $\pi_\beta^{-1}(U_\beta) \cap D \neq \phi$. We now apply **Lemma** 1.48 part (2) to conclude that $\pi_\beta^{-1}(U_\beta) \in \mathscr{D}$.

Now, observe that every basic open set in $X$ containing $x$ is a finite collection of open sets of the form $\pi_{\beta_1}^{-1}(U_{\beta_1}), ...., \pi_{\beta_n}^{-1}(U_{\beta_n})$ (such elements are called *subbasis elements*). So, by part (1) of **Lemma** 1.48, it follows that if $U \subseteq X$ is a basic open set containing $x$ then $U \in \mathscr{D}$.

Finally, if $D \in \mathscr{D}$, $U \subseteq X$ is a basic open set such that $x \in U$, then $D \cap U \neq \phi$ because $D, U \in \mathscr{D}$ and $\mathscr{D}$ has the finite intersection property. So $x \in \overline{D}$ for all $D \in \mathscr{D}$, and this completes our proof. ∎

1.21. **Compactification.** Let $X$ be a topological space. A *compactification* of $X$ is a compact space $Y$ such that $X$ is homeomorphic to a subspace of $X_0$ of $Y$ and $\overline{X_0} = Y$. If $Y$ is a compactification of $X$, then we think of $X$ as a subspace of $Y$.

**Definition 1.30.** Two compactifications $Y_1, Y_2$ of a space $X$ are *equivalent* if there is a homeomorphism $h : Y_1 \to Y_2$ such that $h(x) = x$ for all $x \in X$, i.e there is a homeomorphism between $Y_1, Y_2$ that fixes $X$.

**Example 1.44.** Let $X = (0, 1) \subseteq \mathbb{R}$. Then $Y_1 = [0, 1]$ is a compactification of $X$, and this is easy to see.

Let $Y_2 = S^1 \subseteq \mathbb{R}^2$. We show that $Y_2$ is a compactification of $X$. The map $f : (0, 1) \to S^1$ given by

$$f(x) = (\cos 2\pi x, \sin 2\pi x)$$

is a witness. The image of $(0, 1)$ under this map is $S^1 \setminus \{(1, 0)\}$, and hence $\overline{f((0, 1))} = S^1$.

**Example 1.45.** Let $g : (0, 1) \to \mathbb{R}^2$ be the map given by $g(x) = \left(x, \sin \dfrac{1}{x}\right)$. First, we show that $g$ is a homeomorphism onto its image. Clearly, $g$ is a continuous map. Also, it is one-one because its first component is one-one. Finally, we show that $g$ is a closed map. Let $C$ be any closed subset of $(0,1)$, and let $p$ be a limit point of $g(C)$. We want to show that $p \in g(C)$. So, there is a sequence $x_n$ of points in $g(C)$ such that $x_n \to p$ as $n \to \infty$. Now, write $x_n = (a_n, b_n) \in g(C)$. It is then easily seen that $g(a_n) = (a_n, b_n)$, and each $a_n \in C$. Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the projections. We see that $a_n \to \pi_1(p)$ as $n \to \infty$. Since $C$ is a closed set, it follows that $\pi_1(p) \in C$. Now, consider the point $g(\pi_1(p)) \in g(C)$, and we will show that $p = g(\pi_1(p))$, which will complete our proof. But this is immediate from the continuity of $g$. So, it follows that $g$ is a homeomorphism onto its image.

Now, let $Y_3 = \overline{g((0,1))}$. We show that $Y_3$ is a compactification of $(0,1)$. To be completed.

**Remark 1.49.1.** Try to prove that any two of the above compactifications are equivalent. Try to see if they are even homeomorphic to each other.

**Definition 1.31.** A topological space $X$ is *locally compact at* $x \in X$ if there is some compact subset $C$ of $X$ which contains a neighborhood of $x$. We say that $X$ is *locally compact* if it is locally compact at every point of $X$.

**Example 1.46.** Any compact space is trivially locally compact. It is also easy to see that $\mathbb{R}$ is locally compact. More generally, $\mathbb{R}^n$ is locally compact. $\mathbb{Q}$ is *not* locally compact (this is not a trivial fact). Any ordered topological space with the least upper bound property is locally compact, because of **Theorem** 1.27.

**Theorem 1.50.** *Let $X$ be any space. Then $X$ is a locally compact Hausdorff space if and only if there exists a space $Y$ such that the following hold.*

(1) *$X$ is a subspace of $Y$.*
(2) *$Y - X$ is a singleton.*
(3) *$Y$ is a compact Hausdorff space.*

*If $Y$ and $Y'$ are two spaces satisfying the above conditions, then there is a homeomorphism $h : Y \to Y'$ such that $h(x) = x$ for all $x \in X$, i.e the two compactifications $Y, Y'$ of $X$ are equivalent.*

*Proof.* We will prove this theorem in steps. First the backward direction. So let $Y, Y'$ be two spaces satisfying conditions (1)-(3) of the theorem.

**Step 1:** *$X$ is open in $Y$ and $Y'$.*

Because $Y$ is Hausdorff, $Y - X$ being a singleton is closed in $Y$. Same reasoning holds for $Y'$. This completes the proof.

**Step 2:** Define the map $h : Y \to Y'$ as follows:

$$h(x) = x \qquad \forall x \in X$$
$$h(p) = q \qquad \text{where } Y - X = \{p\},\ Y' - X = \{q\}$$

Now all we need to do is showing that $h$ is a homeomorphism. Clearly, $h$ is a bijective function. So, we only need to show that $h$ is continuous and it is an open map. By symmetry, it suffices to show that for all open sets $U \subseteq Y$, $h(U)$ is open in $Y'$. Let $U \subseteq Y$ be an open set. If $p \notin U$, then $h(U) = U$. Now,

$$U \subseteq Y \text{ is open} \implies U = U \cap X \subseteq X \text{ is open} \implies U \subseteq Y' \text{ is open}$$

and hence $h(U)$ is open in $Y'$ in this case (note that we are using **Step 1** here). Suppose now that $p \in U$. Let $C := Y - U$, which means that $C$ is closed in $Y$. Also note that $C \subseteq X$, and hence $C \subseteq Y, Y'$. Since $Y$ is compact, $C$ being closed is also compact. Because $Y'$ is Hausdorff and $C$ is compact, it follows that $C$ is closed in $Y'$. So

$$h(U) = Y' - C$$

is open in $Y'$. This shows that $h$ is a homeomorphism and completes this part.

**Step 3:** Finally, let us show that $X$ is a locally compact Hausdorff space. $X$ is clearly Hausdorff, since $Y$ is Hausdorff. Now, we prove local compactness. Let $x \in X$, and let $Y - X = \{p\}$. Since $Y$ is Hausdorff, there exist disjoint open sets $U$ and $V$ in $Y$ such that $x \in U, p \in V$. Let $C = Y - V$. Let $C = Y - V$. Then $C \subseteq Y$ is closed, and hence $C$ is compact. Also $C \subseteq X$. So $x \in U \subseteq C \subseteq Y$, and hence $X$ is locally compact.

**Step 4:** Now, we will prove the forward direction of the theorem. Suppose $X$ is locally compact and Hausdorff. We want to construct a space $Y$ with the given properties. Define $Y = X \cup \{\infty\}$, where $\infty$ is just a notation for a new point which is not in $X$. Define the open sets in $Y$ as follows:

- Any open subset $U$ of $X$ is open in $Y$.
- All sets of the form $Y - C$, where $C \subseteq X$ is compact, are open in $Y$.

We claim that this is a topology on $Y$. Clearly, $\phi$ is open in $Y$. Also, note that $Y = Y - \phi$, and hence $Y$ is also open. Now, we will show the closure of open sets under finite intersections.

- If $U_1, U_2 \subseteq X$ are open in $X$, then $U_1 \cap U_2$ is open in $X$, and hence open in $Y$.
- If $C_1, C_2$ are compact subsets of $X$, then

  $$(Y - C_1) \cap (Y - C_2) = Y - (C_1 \cup C_2)$$

  and because $C_1 \cup C_2$ is compact, it follows that this set is open in $Y$.
- Now, suppose $U$ is open in $X$, and $C \subseteq X$ is compact. Then

  $$U \cap (Y - C) = U \cap (X - C)$$

  which is clearly open in $X$, because $X$ is Hausdorff, which implies that $C$ being compact is closed in $X$.

Next, let us show that arbitrary union of open sets is open.

- If $\{U_\alpha\}$ is a collection of open subsets of $X$, then $\bigcup U_\alpha$ is open in $X$, and hence is open in $Y$.
- Let $\{C_\alpha\}$ be a collection of compact subsets of $X$. Then, it is clear that $\bigcap C_\alpha$ is also compact (because $X$ is Hausdorff). So

  $$\bigcup (Y - C_\alpha) = Y - \left(\bigcap C_\alpha\right)$$

  is open in $Y$.
- Finally, let $U$ be an open subset of $X$, and let $C$ be a compact subset of $X$. Then

  $$U \cup (Y - C) = Y - (C - U)$$

  Now $C - U$ is a compact set (since it is a closed subset of $C$), and hence this set is open in $Y$.

So, $Y$ is indeed a topological space. Let us next show that $X$ is a subspace of $Y$. To do this, we need to check that $X \hookrightarrow X \cup \{\infty\} = Y$ is an embedding.

- Let $U \subseteq X$ be open. Then $U \subseteq Y$ is clearly open in $Y$, and hence the inclusion $X \hookrightarrow X \cup \{\infty\}$ is an open map.
- Next, let $U \cap Y$ be an open set. We want to show that $U \cap X$ is open in $X$ (note that $U \cap X$ is the inverse image of $U$ under the inclusion $X \hookrightarrow X \cup \{\infty\}$). If $U$ is such that $U \subseteq X$, then $U$ is already open in $X$. Next, suppose $U = Y - C$ for some compact subset $C \subseteq X$. Then
$$U \cap X = (Y - C) \cap X = X - C$$
is open in $X$, because $C$ is compact and $X$ is Hausdorff. So this shows that $X$ is a subspace of $Y$.

Next, let us show that $Y$ is a compact set. Let $\mathcal{O}$ be an open cover of $Y$. Then one of the elements of the set $\mathcal{O}$ must be of the form $Y - C$ for some compact set $C \subseteq X$. Consider the subset $\mathcal{O}'$ of $\mathcal{O}$ consisting of sets other than $Y - C$. Clearly, $\mathcal{O}'$ is an open cover of $C$. Since $C$ is compact, there are finitely many open sets $U_1, ..., U_n \in \mathcal{O}'$ such that $C \subseteq U_1 \cup ... \cup U_n$. Hence, $\{U_1, ..., U_n, Y - C\}$ is a finite subcover of $Y$. Hence $Y$ is compact.

Finally, let us show that $Y$ is Hausdorff. Let $x, y \in Y$ be distinct points. If $x, y \in X$, then we can separate them, since $X$ is Hausdorff. If $y = \infty$ and $x \in X$, choose a compact set $C \subseteq X$ containing a neighborhood $U$ of $x$ ($U \subseteq X$, and this is where we need local compactness of $X$). Then, $U, Y - C$ are disjoint open sets containing $x$ and $\infty$, respectively. This completes the proof. ■

**Remark 1.50.1.** Let $X$ be a locally compact, Hausdorff space which is *not* compact. Then we claim that $X$ is dense in the space $Y$ constructed in the above theorem. So, it in this, $Y$ is a compactification of $X$. The reasoning is as follows: observe that

$$\begin{aligned} \overline{X} = Y &\iff p \text{ is a limit point of } X \\ &\iff \text{every neighborhood of } p \text{ intersects } X \\ &\iff \{p\} \text{ is not a neighborhood of } p \\ &\iff X \text{ is not compact} \end{aligned}$$

**Definition 1.32.** Let $X$ be a locally compact Hausdorff space which is not compact. The space $Y$ constructed in **Theorem** 1.50 is a compactification of $X$ and is called the *one-point compactification* of $X$.

**Example 1.47.** $S^1$ is the one point compactification of $\mathbb{R}$. $S^2$ is the one-point compactification of $\mathbb{R}^2$ (proof of this to be completed). If we identify $\mathbb{R}^2$ with $\mathbb{C}$, then $S^2 \cong \mathbb{C} \cup \{\infty\}$. This is called the *Riemann Sphere.*

**Lemma 1.51.** *Let $X$ be a Hausdorff space. Then $X$ is locally compact if and only if given $x \in X$ and a neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.*

*Proof.* To be completed. ■

**Proposition 1.52.** *Let $X$ be locally compact Hausdorff; let $A \subseteq X$ be open or closed. Then $A$ is locally compact.*

*Proof.* To be completed. ■

**Proposition 1.53.** *A space $X$ is homeomorphic to an open subspace of a compact Hausdorff space if and only if $X$ is locally compact Hausdorff.*

*Proof.* First, suppose $X$ is locally compact Hausdorff. Let $Y$ be the one-point compactification of $X$ (**Definition** 1.32). Then, $X$ is open in $Y$, and this proves the backward direction of the theorem.

Conversely, suppose $X$ is homeomorphic to an open subsapce of a compact Hausdorff space $Y$, and without loss of generality suppose $X \subseteq Y$. Since $Y$ is compact Hausdorff, **Proposition** 1.52 implies that $X$ is locally compact Hausdorff. This completes the proof. ■

1.22. **Stone-Čech compactification.** The basic question concerning this section will be the following: given a space $X$, can we find a compactification $Y$ of $X$ such that any real valued function $f : X \to \mathbb{R}$ extends to $Y$?

**Lemma 1.54.** *Let $X$ be a space, and let $h : X \to Z$ be an embedding of $X$ into a compact Hausdorff space. Then there is a compactification $Y$ of $X$ which has the following property: there is an embedding $H : Y \to Z$ such that $H(x) = h(x)$ for all $x \in X$. Moreover, $Y$ is uniquely determined upto equivalence.*

*Proof.* Let $X_0 = h(X) \subseteq Z$, and let $Y_0 = \overline{X_0}$. Then $Y_0$ is a compact Hausdorff space, and $Y_0$ is a compactification of $X_0$. Clearly, $Y_0$ is also a compactification of $X$.

Construct a superset $Y$ of $X$ as follows: let $A$ be a set disjoint from $X$ such that there is a bijection $k : A \to Y_0 - X_0$. Let $Y = X \cup A$. Define a function $H : Y \to Y_0$ by:

$$
\begin{aligned}
H(x) &= h(x) \qquad x \in X \\
H(a) &= k(a) \qquad a \in A
\end{aligned}
$$

Clearly, $H$ is a bijection. Give a topology on $Y$ by declaring $U \subseteq Y$ to be open if and only if $H(U) \subseteq Y_0$ is open. Then $H$ is a homeomorphism, and $X$ is a subset of $Y$. Since $Y_0$ is compact Hausdorff, it follows that $Y$ is also compact Hausdorff. Clearly, $Y$ is also a compactification of $X$.

Let us now show that $Y$ is uniquely determined upto equivalence. To be completed. ■

**Theorem 1.55.** *Let $X$ be a completely regular space. Then there is a compactification $Y$ of $X$ such that every bounded continuous map $f : X \to \mathbb{R}$ extends uniquely to a continuous map of $Y$ into $\mathbb{R}$. Further, any such compactification is Hausdorff.*

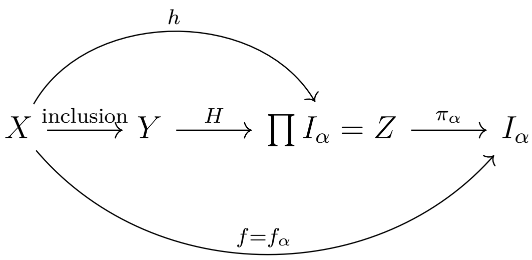
*Proof.* Let $\{f_\alpha\}_{\alpha \in J}$ be the collection of all bounded continuous functions $X \to \mathbb{R}$.

For each $\alpha \in J$, let $I_\alpha := [\inf f_\alpha(x), \sup f_\alpha(x)] \subseteq \mathbb{R}$. Define $h : X \to \prod_{\alpha \in J} I_\alpha = Z$ as

$$h(x) = (f_\alpha(x))_{\alpha \in J}$$

Clearly, $Z$ is Hausdorff. Also, by **Tychonoff's Theorem** 1.49, $Z$ is compact. We claim that $h$ is an embedding. Because $X$ is competely regular, $\{f_\alpha\}$ separates points from closed sets: if $x \in X$ and $A \subseteq X$ is a closed set such that $x \notin A$, then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = \{1\}$. Being bounded, $f \in \{f_\alpha\}$. By **Theorem** 1.41, we see that $h$ is an embedding.

By **Lemma** 1.54, let $Y$ be the compactification of $X$ corresponding to $h : X \to Z$. We claim that $Y$ is the required compactification. Let $f : X \to \mathbb{R}$ be a continuous, bounded map. Then $f = f_\alpha$ for some $\alpha \in J$. Consider the following commutative diagram.



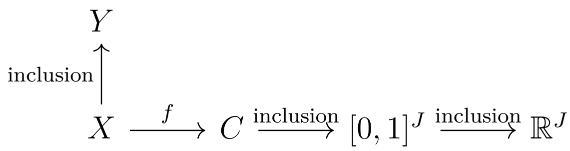

It is easy to argue why the above diagram commutes. Clearly, we see that $H \circ \pi_\alpha$ is the desired continuous extension of $f_\alpha$ to $Y$. The uniqueness of the continuous extension will be proved in the exercise immediately after this theorem. ∎

**Exercise 1.16.** Let $X$ be a space and let $A \subseteq X$. Let $f : A \to Z$ be a continuous map where $Z$ is Hausdorff. Then there exists *at most* one extension of $f$ to a continuous function $g : \overline{A} \to Z$.

**Solution.** To be completed. Idea: If there are two extensions $f, g : \overline{A} \to Z$. Say there is some $a \in \overline{A}$ such that $f(a) \neq g(a)$. Now use the Hausdorffness of $Z$ to get disjoint open sets $U, V$ such that $f(a) \in U$ and $g(a) \in V$. Then, take the sets $f^{-1}(U)$ and $g^{-1}(V)$; both of these sets contain the point $a$. Since $a \in \overline{A}$, every open neighborhood of $a$ will intersect $A$. Find a contradiction from this.

**Theorem 1.56.** *Let $X$ be a completely regular space, and let $Y$ be a compactification satisfying the extension property proved in the previous theorem. Let $C$ be any compact Hausdorff space and let $f : X \to C$ be continuous. Then $f$ extends uniquely to a continuous map $g : Y \to C$.*

*Proof.* Since $C$ is compact Hausdorff, it is normal by **Theorem** 1.38, and hence by **Urysohn's Lemma** 1.39 $C$ is completely regular. Then, just like we saw in the proof of **Theorem** 1.55, we can embed $C$ inside $[0,1]^J$, where $J$ is the cardinality of the set of all bounded continuous functions $f : C \to \mathbb{R}$. So, without loss of generality we assume $C \subseteq [0,1]^J$. Now consider the following diagram.




The above diagram gives us a map from $X$ to $\mathbb{R}^J$, i.e we $J$ coordinate maps from $X$ to $\mathbb{R}$. Also, each of these maps is bounded. So, by **Theorem** 1.55, each coordinate map extends uniquely to $Y$, and hence using these coordinate extensions, we can extend the map $f$ to $Y$. Let the extended map be $g$. So, $g : Y \to \mathbb{R}^J$. We claim that $g(Y) \subseteq C$. This is true because

$$g(Y) = g(\overline{X}) \subseteq \overline{g(X)} = \overline{f(X)} \subseteq \overline{C} = C$$

This completes the proof. ∎

**Theorem 1.57.** *Let $X$ be completely regular. If $Y_1$ and $Y_2$ are two compactifications satisfying the extension property in the above theorem, then $Y_1$ and $Y_2$ are equivalent.*

*Proof.* This is a good exercise, and the idea is the following: we know that $Y_1$ and $Y_2$ are compact Hausdorff, which was guaranteed by **Theorem** 1.55. Then, just invoke **Theorem** 1.56 on the inclusion maps $X \hookrightarrow Y_1$ and $X \hookrightarrow Y_2$ to get maps from $Y_1$ to

$Y_2$ and $Y_2$ to $Y_1$ which restrict to the identity on $X$, and which are inverses of each other. ■

**Definition 1.33.** Let $X$ be a completely regular space. The compactification $Y$ of $X$ satisfying the extension property of **Theorem** 1.55 is called the *Stone-Čech compactification* of $X$.

## 2. Algebraic Topology

2.1. **Homotopy of paths.** Let $f, f' : X \to Y$ be continuous maps, where $X, Y$ are arbitrary topological spaces. We say that $f$ *is homotopic to* $f'$ if there is a continuous function $F : X \times I \to Y$ such that

$$F(x,0) = f(x), \quad F(x,1) = f'(x) \qquad \forall\ x \in X$$

Here $I = [0,1]$ is the unit interval. In this case, we use the notation $f \sim f'$.

**Definition 2.1.** If $f : X \to Y$ is homotopic to a constant function $c : X \to Y$ then $f$ is said to be *null-homotopic*.

**Definition 2.2.** A *path* is a continuous map $f : I \to X$. Two paths $f, f'$ in $X$ are said to be *path-homotopic* if they have the same initial point $x_0$ and the same final point $x_1$ and if there is a continuous map $F : I \times I \to X$ such that for all $s, t \in I$ we have

$$F(s,0) = f(s) \quad , \quad F(s,1) = f'(s)$$
$$F(0,t) = x_0 \quad , \quad F(1,t) = x_1$$

So, $F$ is really a homotopy that fixes two endpoints. In this case, we use the notation $f \sim_p f'$.

**Lemma 2.1** (**Pasting Lemma**). *Let $f : X \to Y$ be a map such that $X = A \cup B$, where $A, B$ are closed subsets of $X$. $f|_A$ and $f|_B$ are continuous. Then $f$ is also continuous.*

*Proof.* Let $C$ be a closed subset of $Y$. We want to show that $f^{-1}(C)$ is closed in $X$. By hypothesis, we know that $A \cap f^{-1}(C)$ is closed in $A$. This means that $A \cap f^{-1}(C)$ is also closed in $X$ (because $A$ is closed). Similarly, $f^{-1}(C) \cap B$ is closed in $X$. So,

$$(f^{-1}(C) \cap A) \cup (f^{-1}(C) \cap B) = f^{-1}(C)$$

is closed in $X$, completing the proof. ■

**Lemma 2.2.** *The relations $\sim$ and $\sim_p$ are equivalence relations.*

*Proof.* Suppose $f_1, f_2, f_3 : X \to Y$ are continuous maps, and suppose $f_1', f_2', f_3' : I \to X$ are paths. It is clear that $f_1 \sim f_1$ and $f_1' \sim f_1'$, by taking the constant homotopies. So $\sim$ and $\sim_p$ are reflexive relations.

Next, assume $f_1 \sim f_2$ (or $f_1' \sim_p f_2'$). So, there exists a homotopy $F : X \times I \to Y$ (or a path homotopy $F : I \times I \to X$) between $f_1$ and $f_2$ (or $f_1'$ and $f_2'$). Consider the map

$$G(x,t) = F(x, 1-t)$$

and it can be easily verified that $G$ is a homotopy between $f_2$ and $f_1$ (or a path homotopy between $f_2'$ and $f_1'$), and this shows that $\sim$ and $\sim_p$ are symmetric relations. The proof that $\sim$ and $\sim_p$ are transitive makes up a good exercise. As a hint, consider using the **Pasting Lemma** 2.1. ■

**Definition 2.3.** If $f$ is a path, then we denote its equivalent class under $\sim_p$ by $[f]$.

**Definition 2.4.** Let $X$ be any space. If $f$ is a path in $X$ from $x_0$ to $x_1$, and if $g$ is a path from $x_1$ to $x_2$, then the product $f * g$ of $f$ and $g$ is a path $h : I \to X$ defined as

$$h(s) = \begin{cases} f(2s) & s \in [0, 1/2] \\ g(2s-1) & s \in [1/2, 1] \end{cases}$$

The path $f * g$ is called the **concatenation** of $f$ and $g$.

**Lemma 2.3.** *If $f \sim_p f'$ and $g \sim_p g'$ then $f * g \sim_p f' * g'$, assuming both the concatenations are defined.*

*Proof.* Let $x_0$, $x_1$ and $x_2$ in $X$ be the starting point of $f, f'$, starting point of $g, g'$ and the ending point of $g, g'$ respectively. Observe that

$$(f * g)(0) = x_0 \quad , \quad (f' * g')(0) = x_0$$
$$(f * g)(1) = x_2 \quad , \quad (f' * g')(1) = x_2$$

and this means that both paths $f * g$ and $f' * g' *$ have the same starting and ending points.

Now, let $F, G$ be path homotopies between $f, f'$ and $g, g'$ respectively. Define the map $H : I \times I \to X$ by the following.

$$H(s,t) = \begin{cases} F(2s,t) & , \quad s \in [0, 1/2] \\ G(2s-1,t) & , \quad s \in [1/2, 1] \end{cases}$$

Observe that $[0,1] \times [0,1]$ can be written as a union of the two closed sets $[0, 1/2] \times [0, 1]$ and $[1/2, 1] \times [0, 1]$. By the **Pasting Lemma** 2.1, it is clear that $H$ is a continuous map. It is then an easy check that $H$ is a path homotopy between $f * g$ and $f' * g'$. ■

**Remark 2.3.1.** So, it follows that $*$ is well-behaved under equivalence classes of paths. So we defined $[f] * [g] = [f * g]$, given that $f(1) = g(0)$.

**Proposition 2.4.** *Let $k : X \to Y$ be a continuous map. Let $F$ be a path homotopy between two paths $f, f'$ in $X$. Then $k \circ F$ is a path homotopy between $k \circ f$ and $k \circ f'$.*

*Proof.* Let $F : I \times I \to X$ be a path homotopy between paths $f, f'$ in $X$. Clearly, $k \circ F$ is a continuous map. Now, suppose $x_0 = f(0) = f'(0)$ and $x_1 = f(1) = f'(1)$. We know that

$$F(0,t) = x_0 \quad , \quad F(1,t) = x_1 \qquad t \in I$$

Then, we have

$$k \circ F(0,t) = k(x_0) \quad , \quad k \circ F(1,t) = k(x_1) \qquad t \in I$$

which implies that $k \circ F$ is a path homotopy between $k \circ f$ and $k \circ f'$. ■

**Proposition 2.5.** *Let $k : X \to Y$ be a continuous map. Let $f, g$ be paths in $X$ such that $f(1) = g(0)$. Then*

$$k \circ (f * g) = (k \circ f) * (k \circ g)$$

*Proof.* This is an easy computation. Note that

$$k \circ (f * g)(s) = \begin{cases} k \circ f(2s) & , \quad 0 \le s \le 1/2 \\ k \circ g(2s-1) & , \quad 1/2 \le s \le 1 \end{cases}$$

and the above is the same as the path $(k \circ f) * (k \circ g)$. This completes the proof. ■

**Theorem 2.6.** *Let $X$ be any topological space. The operation $*$ on paths in $X$ has the following three properties.*

(1) *(Associativity) If* $[f]*([g]*[h])$ *is defined, then so is* $([f]*[g])*[h]$*, and they are equal.*
(2) *(Right and left identities) Given* $x \in X$*, let* $e_x : I \to X$ *denote the constant path at* $x$*. If* $f$ *is a path in* $X$ *from* $x_0$ *to* $x_1$*, then*

$$[f]*[e_{x_1}] = [f]$$

*and*

$$[e_{x_0}]*[f] = [f]$$

(3) *(Inverse) Given a path in* $X$ *from* $x_0$ *to* $x_1$*, let* $\overline{f} : I \to X$ *be the path defined by*

$$\overline{f} : I \to X \quad , \quad \overline{f}(s) = f(1-s)$$

*Then* $[f]*[\overline{f}] = [e_{x_0}]$ *and* $[\overline{f}]*[f] = [e_{x_1}]$*.*

*Proof.* We will be using the fact that any two paths in $I$ with the same initial and same final point are path homotopic, and this is true because $I$ is a convex subset of $\mathbb{R}$.

First, let us prove (2). Let $e_0$ denote the constant path in $I$ at 0. Let $i : I \to I$ be the identity map. So $i$ is a path in $I$ from 0 to 1. Since $I \subseteq \mathbb{R}$ is a convex set, there is a path homotopy $G$ in $I$ between $i$ and $e_0 * i$, i.e

$$I \times I \xrightarrow{G} I \xrightarrow{f} X$$

By **Proposition** 2.4, $f \circ G$ is a path homotopy between $f \circ i = f$ and $f \circ (e_0 * i)$. By **Proposition** 2.5 we see that

$$f \circ (e_0 * i) = (f \circ e_0) * (f \circ i) = e_{x_0} * f$$

This means that

$$f \sim_p e_{x_0} * f$$

which implies that $[e_{x_0}]*[f] = [f]$. The right identity is similarly proven.

Next, we prove (3). So let $f$ be a path in $X$ from $x_0$ to $x_1$. Let $\overline{f}$ be the path defined by

$$\overline{f}(s) = f(1-s)$$

Define the map $\overline{i} : I \to I$ by $\overline{i}(s) = 1-s$. Then $i * \overline{i}$ is a path in $I$ from 0 to 0. Since $I$ is convex we see that $i * \overline{i} \sim_p e_0$. So suppose $H$ is a path homotopy between $i * \overline{i}$ and $e_0$. So by **Proposition** 2.4 $f \circ H$ is a path homotopy between $f \circ (i * \overline{i})$ and $f \circ e_0 = e_{x_0}$. By **Proposition** 2.5 we have

$$f \circ (i * \overline{i}) = (f \circ i) * (f \circ \overline{i}) = f * \overline{f}$$

and hence it follows that $f * \overline{f} \sim_p e_{x_0}$, which implies that $[f]*[\overline{f}] = [e_{x_0}]$. Similarly, we can show that $[\overline{f}]*[f] = [e_{x_1}]$, and this completes the proof.

Finally, we prove (1). Suppose $[f]*([g]*[h])$ is well defined. This means that $f(1) = g(0)$ and $g(1) = h(0)$. Then, observe that $([f]*[g])*[h]$ is also well-defined.

Observe that we have the following, which follow from the definition.

$$f * (g * h)(s) = \begin{cases} f(2s) & , \quad s \in [0, 1/2] \\ g(2(2s-1)) & , \quad s \in [1/2, 3/4] \\ h(2(2s-1)-1) & , \quad s \in [3/4, 1] \end{cases}$$

$$(f * g) * h(s) = \begin{cases} f(2(2s)) & , \quad s \in [0, 1/4] \\ g(2(2s)-1)) & , \quad s \in [1/4, 1/2] \\ h(2s-1) & , \quad s \in [1/2, 1] \end{cases}$$

We can clearly see that they are not equal. Here is a general fact that we will use: if $[a, b]$, $[c, d]$ are two intervals in $\mathbb{R}$, then there is a unique map $p : [a, b] \to [c, d]$ of the form $p(x) = mx + k$ such that $p(a) = c$ and $p(b) = d$, i.e $p$ is a positive linear map. Now, given $0 < a < b < 1$, we define a path $k_{a,b}$ in $X$ as follows:

- On $[0, a]$, $k_{a,b} =$ the positive linear map of $[0, a]$ to $I$ followed by $f$
$$[0, a] \xrightarrow{p} [0, 1] \xrightarrow{f} X$$
i.e $k = f \circ p$.
- On $[a, b]$, we define $k_{a,b} =$ the positive linear map of $[a, b]$ to $I$ followed by $g$
$$[a, b] \xrightarrow{p} [0, 1] \xrightarrow{g} X$$
i.e $k_{a,b} = g \circ p$.
- On $[b, 1]$, we define $k_{a,b} =$ the positive linear map of $[b, 1]$ to $I$ followed by $h$
$$[b, 1] \xrightarrow{p} [0, 1] \xrightarrow{h} X$$
i.e $k_{a,b} = h \circ p$.

We claim that for $0 < a < b < 1$ and $0 < c < d < 1$, the two paths $k_{a,b}$ and $k_{c,d}$ in $X$ are path homotopic. We will use the fact that $I$ is convex. Let $p : I \to I$ be the map obtained by pasting the three positive linear maps: $[0, a]$ to $[0, c]$, $[a, b]$ to $[c, d]$ and $[b, 1]$ to $[d, 1]$. Since $I$ is convex, $p$ and $i$ are path homotopic (since they have the same endpoints). Let $P$ be the path homotopy between $p$ and $i$ in $I$. Then by **Proposition** 2.4 $k_{c,d} \circ P$ is a path homotopy in $X$ between $k_{c,d} \circ p$ and $k_{c,d} \circ i$. Now clearly we see that $k_{c,d} \circ i = k_{c,d}$. We claim that

$$k_{c,d} \circ p = k_{a,b}$$

but this is actually straightforward. So this means that $k_{a,b} \sim_p k_{c,d}$, and this proves the claim. We are now done, because

$$f * (g * h) = k_{1/2,3/4}$$
$$(f * g) * h = k_{1/4,1/2}$$

■

**Definition 2.5.** Let $X$ be a topological space. A *loop* at $x \in X$ is a path in $X$ that begins and ends at $x$. The set of path homotopy classes of loops at $x$ is denoted by $\pi_1(X, x)$. Then $*$ is a binary operation in $\pi_1(X, x)$. Further, by the above theorem, $*$ is associative, and the constant loop $[e_x]$ is the identity of this operation. Hence, $\pi_1(X, x)$ is a group, called the *fundamental group* of $X$ relative to the *base point* $x$.

**Example 2.1.** Let $X = \mathbb{R}^n$. Then $\pi_1(X, x)$ is the trivial group for all $x \in X$. This follows because any two paths in $\mathbb{R}^n$ are path homotopic, if they share the same endpoints (just use the straight line homotopy). More generally, $\pi_1(X, x)$ is the trivial group if $X \subseteq \mathbb{R}^n$ is a convex set.

**Definition 2.6.** Let $X$ be a space and $x_0, x_1 \in X$. Suppose $\alpha$ is a path in $X$ from $x_0$ to $x_1$. Then we define a map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$[f] \mapsto [\overline{\alpha} * f * \alpha]$$

Clearly, $\hat{\alpha}$ is well-defined on homotopy classes because $*$ is so.

**Theorem 2.7.** *$\hat{\alpha}$ is a group isomorphism.*

*Proof.* First, we have

$$\begin{aligned}\hat{\alpha}([f] * [g]) &= [\overline{\alpha} * f * g * \alpha] \\ &= [\overline{\alpha} * f * \alpha * \overline{\alpha} * g * \alpha] \\ &= [\overline{\alpha} * f * \alpha] * [\overline{\alpha} * g * \alpha] \\ &= \hat{\alpha}([f]) * \hat{\alpha}([g])\end{aligned}$$

and hence $\hat{\alpha}$ is a group homomorphism. Next, let $\beta = \overline{\alpha}$. Then we have a group homomorphism $\hat{\beta} : \pi_1(X, x_1) \to \pi_1(X, x_0)$ given by

$$[h] \mapsto [\overline{\beta} * h * \beta]$$

Now, it is an easy check that $\hat{\alpha} \circ \hat{\beta} = \mathrm{id}_{\pi_1(X, x_1)}$ and that $\hat{\beta} \circ \hat{\alpha} = \mathrm{id}_{\pi_1(X, x_0)}$, and this implies that $\hat{\alpha}$ is a group isomorphism. ■

**Corollary 2.7.1.** *If $X$ is path-connected and $x_0, x_1 \in X$ then*

$$\pi_1(X, x_0) \cong \pi_1(X, x_1)$$

*So if $X$ is path-connected, we can speak of the fundamental group of $X$ without reference to the base point.*

**Definition 2.7.** A space $X$ is said to be *simply connected* if it is path connected and $\pi_1(X, x)$ is trivial for all $x \in X$.

**Lemma 2.8.** *In a simply connected space $X$, any two paths having common endpoints are path homotopic.*

*Proof.* Let $f, g$ be two paths in $X$ having the same endpoints. Then, $f * \overline{g}$ is a loop at some point $x_0 \in X$. Since $X$ is simply connected, we know that $f * g \sim_p e_{x_0}$. So, we get

$$\begin{aligned}&f * \overline{g} * g \sim_p e_{x_0} * g \\ \Longrightarrow\ &f * e_{x_1} \sim_p g \\ \Longrightarrow\ &f \sim_p g\end{aligned}$$

and this completes the proof. ■

**Definition 2.8.** Let $h : (X, x) \to (Y, y)$ be a continuous map of topological spaces such that $h(x) = y$. Define a map $h_* : \pi_1(X, x) \to \pi_1(Y, y)$ as follows:

$$[f] \mapsto [h \circ f]$$

**Proposition 2.9.** *$h_*$ is a well-defined map which is a group homomorphism.*

*Proof.* First, we show the well-definedness of $h$. Let $f \circ_p f'$, where $f, f'$ are loops in $X$ based at $x$. Let $F$ be a path homotopy between $f$ and $f'$. By **Proposition** 2.4, $h \circ F$ is a path homotopy between $h \circ f$ and $h \circ f'$, and hence $h$ is a well defined map.

Next, we show that $h_*$ is a group homomorphism. We have

$$\begin{aligned} h_*([f] * [g]) &= h_*([f * g]) \\ &= [h \circ (f * g)] \\ &= [(h \circ f) * (h \circ g)] \\ &= h_*([f]) * h_*([g]) \end{aligned}$$

and this completes the proof. ∎

**Theorem 2.10.** *Let $h : (X, x) \to (Y, y)$ and $k : (Y, y) \to (Z, z)$ be continuous maps. Then*

$$(k \circ h)_* = k_* \circ h_*$$

*Further, if $i : (X, x) \to (X, x)$ is the identity map, then $i_*$ is the identity homomorphism.*

*Proof.* The proof is simple. Observe that

$$\begin{aligned} (k_* \circ h_*)[f] &= k_*(h_*[f]) \\ &= k_*([h \circ f]) \\ &= [k \circ (h \circ f)] \\ &= [(k \circ h) \circ f] \\ &= (k \circ h)_*[f] \end{aligned}$$

Also, we see that

$$i_*([f]) = [i \circ f] = [f]$$

∎

**Corollary 2.10.1.** *If $h : (X, x) \to (Y, y)$ is a homeomorphism, then $h_* : \pi_1(X, x) \to \pi_1(Y, y)$ is a group isomorphism.*

*Proof.* This is a direct consequence of **Theorem** 2.10. Consider

$$(X, x) \xrightarrow{h} (Y, y) \xrightarrow{h^{-1}} (X, x)$$

and use the previous theorem. ∎

**Corollary 2.10.2.** *If $X, Y$ are path connected, and for some $x \in X, y \in Y$, $\pi_1(X, x)$ is not isomorphic to $\pi_1(Y, y)$, then $X$ and $Y$ are not homeomorphic.*

2.2. **Covering Spaces.** Let $p : E \to B$ be a continuous, surjective map. An open set $U \subseteq B$ is said to be *evenly covered by $p$* if $p^{-1}(U)$ can be written as a disjoint union of open sets $V_\alpha \subseteq E$ such that $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism for all $\alpha$. The collection $\{V_\alpha\}$ is called a *partition* of $p^{-1}(U)$ into *slices*.

**Definition 2.9.** Let $p : E \to B$ be a continuous, surjective map. If every point of $B$ has a neighborhood $U$ which is evenly covered by $p$, then $p$ is called a *covering map* and $E$ is called a *covering space* of $B$.

**Example 2.2.** The identity map $i : X \to X$ is a covering map (in this case, there is only 1 slice). The map $p : X \times \{1, ..., n\} \to X$ given by $p(x, i) = x$ for all $i, x$ is a covering map, where $\{1, ..., n\}$ is given the discrete topology. This map has $n$ slices.

**Proposition 2.11.** *If $p : E \to B$ is a covering map and $b \in B$ then $p^{-1}(b) \subseteq E$ has the discrete topology.*

*Proof.* Suppose $b \in B$, and let $b \in U$ be an evenly covered neighborhood of $b$. Let $p^{-1}(U) = \bigcup_\alpha V_\alpha$ be a partition into slices. Since $V_\alpha \subseteq E$ is open and $V_\alpha \cap p^{-1}(b)$ is a singleton, each point in $p^{-1}(b)$ is open in $p^{-1}(b)$. So, $p^{-1}(b)$ has the discrete topology. ■

**Proposition 2.12.** *If $p : E \to B$ is a covering map, then $p$ is open.*

*Proof.* Let $v \subseteq E$ be an open set and let $x \in p(V)$. Let $U$ be an evenly covered neighborhood of $x$. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Let $y \in V$ be such that $p(y) = x$. Say $y \in V_\beta$ for some $\beta$. Then $V_\beta \cap V$ is open in $V_\beta$. Now $p(V_\beta \cap V) \subseteq U$ is open since $p|_{V_\beta} : V_\beta \to U$ is a homeomorphism. Since $U$ is open in $B$ and $p(V_\beta \cap V) \subseteq U$ is open in $U$, we see that $p(V_\beta \cap V)$ is open in $B$. Observe that $x \in U \cap p(V_\beta \cap V)$. So, we have found an open neighborhood $p(V_\beta \cap V)$ of $x$ contained in $p(V)$. Hence, $p(V) \subseteq B$ is open. ■

**Proposition 2.13.** *If $p : E \to B$ is a covering map, then $p$ is a local homeomorphism, i.e each point of $E$ has a neighborhood that is mapped homeomorphically by $p$.*

*Proof.* Let $e \in E$ and $b = p(e) \in B$. Let $U$ be an evenly covered neighborhood of $b$. Say $p^{-1}(U) = \bigcap_\alpha V_\alpha$ is a partition into slices. Then $e \in V_\alpha$ for some $\alpha$, and $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Note that each $V_\alpha \subseteq E$ is open, and this completes the proof. ■

**Theorem 2.14.** *The map $p : \mathbb{R} \to S^1$ given by $x \mapsto (\cos 2\pi x, \sin 2\pi x)$ is a covering map.*

*Proof.* It is clear that $p$ is continuous and surjective. Let

$$U := \{(\cos 2\pi x, \sin 2\pi x) \in S^1 \mid \cos 2\pi x > 0\}$$

Then

$$p^{-1}(U) = \{x \in \mathbb{R} \mid \cos 2\pi x > 0\} = \bigcup_{n \in \mathbb{Z}} V_n$$

where $V_n := (n - 1/4, n + 1/4)$. We claim that $U$ is evenly covered by $p$. Clearly, each $V_n$ is open in $\mathbb{R}$, and if $m \neq n$ we have that $V_n \cap V_m = \phi$. It remains to show that $p|_{V_n} : V_n \to U$ is a homeomorphism.

First, note that $p|_{\overline{V_n}} : [n - 1/4, n + 1/4] \to \overline{U}$ is bijective: it is one-one because $\sin 2\pi x$ is monotonically increasing on $\overline{V_n}$. It is onto because $p(n - 1/4) = (-1, 0)$, $p(n + 1/4) = (1, 0)$, and then we can just use the intermediate value theorem.

Now, it is clear that $p|_{\overline{V_n}}$ is continuous. Since $\overline{V_n}$ is compact and $\overline{U}$ is Hausdorff, it follows that $p|_{\overline{V_n}}$ is a homeomorphism. Hence $p|_{V_n} : V_n \to U$ is also a homeomorphism.

A similar argument shows that all the other open half circles are evenly covered (note that $U$ is an open half circle). So, $p : \mathbb{R} \to S^1$ is a covering map. ■

**Theorem 2.15.** *Consider $S^1 = \{\cos\theta + i \sin\theta \mid \theta \in [0, 2\pi)\}$ as a subspace of $\mathbb{C}$. The map $p : S^1 \to S^1$ given by $p(z) = z^2$ is a covering map.*

*Proof.* Let $z \in S^1$, and we will write $z = e^{i\theta} = \cos\theta + i\sin\theta$. Then $p(z) = z^2 = e^{2i\theta} = \cos 2\theta + i \sin 2\theta$. Let $U = S^1 - \{1\}$. Then $U \subseteq S^1$ is open. Also,

$$p^{-1}(U) = \{e^{i\theta}, \theta \in [0, 2\pi) \mid (\cos 2\theta, \sin 2\theta) \neq (1, 0)\} \cong (0, \pi) \cup (\pi, 2\pi) = V_1 \cup V_2$$

It is easy to check that $p|_{V_i} : V_i \to U$ are homeomorphisms. So $U = S^1 - \{1\}$ is evenly covered. Similarly $S^1 - \{-1\}$ is evenly covered. So $p$ is a covering map. ■

**Exercise 2.1.** Show that the map $p : S^1 \to S^1$ given by $p(z) = z^n$ is a covering map for all $n \geq 1$. Show that there are $n$ slices.

**Solution.** To be completed.

**Example 2.3.** We now see an example of a map which is not a covering map. Let $p : \mathbb{R}_+ \to S^1$ be the map $x \mapsto (\cos 2\pi x, \sin 2\pi x)$. We show that $p$ is not a covering map. To show this, we show that $b_0 = (1, 0)$ has not evenly covered neighborhood. If $U$ is any neighborhood of $b_0$, then $p^{-1}(U)$ is a union of small neighborhoods $V_n$ for $n > 0$ and $V_0$ of the form $V_0 = (0, \epsilon)$ for some $\epsilon > 0$. Complete this example!

**Proposition 2.16.** *Let $p : E \to B$ be a covering map. If $B_0$ is a subspace of $B$ and $E_0 = p^{-1}(B_0)$ then $p_0 = p|_{E_0} : E_0 \to B_0$ is a covering map.*

*Proof.* For $b \in B_0$, let $U \subseteq B$ be a neighborhood of $b$ which is evenly covered by $p$. Say $p^{-1}(U) = \bigcup_\alpha V_\alpha$. Then $U \cap B_0$ is a neighborhood of $b$ in $B_0$ which is evenly covered by $p_0$: $p_0(U \cap B_0) = \bigcup_\alpha V_\alpha \cap E_0$. This completes the proof. ∎

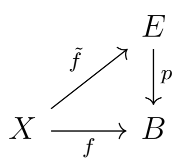
**Proposition 2.17.** *Let $p : E \to B$ and $p' : E \to B$ be covering maps. Then, the map $p \times p' : E \times E' \to B \times B'$ given by $(e, e') \mapsto (p(e), p'(e'))$ is also a covering map.*

*Proof.* Let $b \in B, b' \in B'$ be points. Let $U, U'$ be evenly covered neighborhoods of $b, b'$ in $B, B'$ respectively. Let $p^{-1}(U) = \bigcup_\alpha V_\alpha$ and let $p^{-1}(U') = \bigcup_\beta V'_\beta$. Then

$$p^{-1}(U \times U') = \bigcup_{\alpha,\beta} V_\alpha \times V_{\beta'}$$

is a partition of $p^{-1}(U \times U')$ into slices. This proves the claim. ∎

2.3. **Lifting of Paths.** Let $p : E \to B$ be a continuous map. If $f : X \to B$ is a continuous map, a *lifting* of $f$ is a continuous map $\tilde{f} : X \to E$ such that $p \circ \overline{f} = f$. We will be most interested in lifting paths in $B$ to $E$ (so we usually take $X = [0, 1]$) when $p$ is a covering map. This is represented via the following diagram.



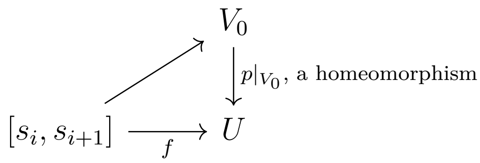

**Lemma 2.18.** *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Any path $f : I \to B$ beginning at $b_0$ has a unique lifting to a path $\tilde{f} : I \to E$ beginning at $e_0$.*

*Proof.* First we cover $B$ by evenly covered neighborhoods $\{U_\alpha\}$. Then, choose a subdivision of $[0, 1]$: $s_0 = 0 < s_1 < \cdots < s_n = 1$ such that $f([s_i, s_{i+1}])$ is contained in an evenly covered neighborhood $U$. This can be done as follows: clearly, $\{f^{-1}(U_\alpha)\}$ is an open cover of $I$. Since $I$ is compact, by the **Lebesgue Number Lemma** 1.30 there is some $\delta > 0$ such that for each subset $A \subseteq I$ of diameter $\leq \delta$, there is some $f^{-1}(U_\alpha)$ such that $A \subseteq f^{-1}(U_\alpha)$. So, we see that

$$[0, s_1] \cup [s_1, s_2] \cup \cdots \cup [s_{n-1}, 1] = I$$

Define $\tilde{f}(s_0) = e_0$. By induction, suppose $\tilde{f}$ is defined on $[0, s_i]$. Next, we define $\tilde{f}$ on $[s_i, s_{i+1}]$ as follows. Let $U \subseteq B$ be an evenly covered neighborhood containing $f([s_i, s_{i+1}])$. Let $p^{-1}(U) = \bigcup V_\alpha$ be a partition into slices. Then $f(s_i) = p(\tilde{f}(s_i)) \in U$, which implies that $\tilde{f}(s_i) \in \bigcup V_\alpha$. Since the slices are disjoint, there exists exactly one slice, say $V_0$, containing $\tilde{f}(s_i)$.

Now, we use the homeomorphism $p|_{V_0} : V_0 \xrightarrow{\sim} U$. We have the following diagram.




We need to define a map from $[s_i, s_{i+1}]$ to $V_0$ so that this diagram commutes. So, we can just define $\tilde{f} : [s_i, s_{i+1}] \to V_0$ as

$$\tilde{f}(s) = (p|_{V_0})^{-1}(f(s))$$

and clearly $\tilde{f}$ is continuous.

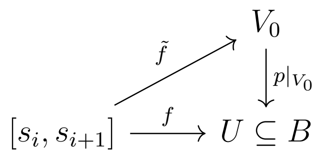
Proceeding this way, we define a continuous$\tilde{f} : [0,1] \to E$ and by construction, it is a lifting of $f$. This completes the proof of existence.

Now, let us prove uniqueness of the lifting. Let $\tilde{f}'$ be another lifting of $f$ starting at $e_0$. So, we know that

$$\tilde{f}'(0) = \tilde{f}(0) = e_0$$

and that $p \circ \tilde{f}' = p \circ \tilde{f} = f$.

Now, suppose $\tilde{f} = \tilde{f}'$ on $[0, s_i]$ (we have shown this for $i = 0$). Let $V_0 \subseteq E$ be an open set as above (in the construction of $\tilde{f}$) such that $p|_{V_0} : V_0 \to U$ is a homeomorphism and $\tilde{f}(s_i) = \tilde{f}'(s_i) \in V_0$. So, we have following diagram.




Now, since $\tilde{f}'$ is a lifting of $f$, we must have

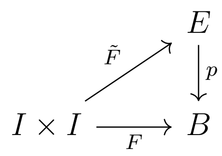
$$\tilde{f}'([s_i, s_{i+1}]) \subseteq p^{-1}(U) = \bigcup V_\alpha$$

On the other hand, slices are open, disjoint and $[s_i, s_{i+1}$ is connected. So $\tilde{f}'([s_i, s_{i+1}])$ is contained in a single slice. But that slice must be $V_0$, since $\tilde{f}'(s_i) \in V_0$. Then, since $\tilde{f}'$ is a lifting, we see that

$$\tilde{f}'(s) = (p|_{V_0})^{-1}(f(s)) = \tilde{f}(s)$$

for all $s \in [s_i, s_{i+1}]$. Proceeding this way, we can conclude that $s \in [s_i, s_{i+1}]$. This completes the proof. ■

**Lemma 2.19.** *Let $p : E \to B$ be a covering map and let $p(e_0) = b_0$. Let $F : I \times I \to B$ be a continuous map with $F(0,0) = b_0$. Then there is a unique lifting $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0,0) = e_0$. If $F$ is a path homotopy, then so is $\tilde{F}$.*




*Proof.* Proof is essentially the same as of **Lemma** 2.18. First, we use **Lemma** 2.18 to lift $F|_{0\times I}$ and $F|_{I\times 0}$ (uniquely). Here $I \times 0$ is the bottom edge of the unit square, and $0 \times I$ is the vertical edge of the square.

Now, choose subdivisions $0 = s_0 < s_1 < \cdots < s_n = 1$ an $0 = t_0 < t_1 < \cdots < t_m = 1$ such that

$$F(I_i \times J_j) \subseteq \text{some evenly covered neighborhood in } B$$

where $I_i = [s_{i-1}, s_i]$ and $J_j = [t_{j-1}, t_j]$, and we do this by invoking the **Lebesgue Number Lemma** 1.30, just like we did in **Lemma** 2.18. Imagine the rectangles $I_i \times J_j$ to form a grid of the unit square.

We now define $\tilde{F}$ step by step: first define for all squares $I_i \times J_1$, $0 \le i \le n$, then for all squares $I_i \times J_2$, $0 \le i \le n$ and so on. This can be thought of as starting at the bottom most square of the grid, finishing the bottom most row, then moving to the second row, and so on.

Say $\tilde{F}$ is defined on all squares *before* $I_{i_0} \times J_{j_0}$, i.e for all squares $I_i \times J_j$ for

(1) $j < j_0$ and
(2) $j = j_0$, $i < i_0$.

Let $A =$ union of $I \times 0$, $0 \times I$ and all squares before $I_{i_0} \times J_{j_0}$. Then,

$$C := A \cap (I_{i_0} \times J_{j_0})$$

is the union of the bottom and left edges of the square $I_{i_0} \times J_{j_0}$. Now, $\tilde{F}$ is defined on $C$, and $C$ is connected. We also know that $F(I_{i_0} \times J_{j_0})$ is contained in some evenly covered neighborhood $U$ of $B$. Hence, by the connectedness of $C$,

$$\tilde{F}(C) \subseteq \text{a single slice of } U, \text{ say } V_0$$

So, as in **Lemma** 2.18, extend $\tilde{F}$ to $I_{i_0} \times J_{j_0}$ by

$$\tilde{F}(x) := (p|_{V_0})^{-1}(F(x))$$

We continue this way to obtain $\tilde{F} : I \times I \to E$ which is a continuous lifting of $F$.

The proof of uniqess is exactly the same as in **Lemma** 2.18. We are given $\tilde{F}(0,0) = e_0$. We proceed step by step to prove uniqueness. Since $C$ (constructed above) is connected, any lifting of $F$ must map a subsquare $I_i \times J_j$ to a single slice. Then we conclude that there is only once choice for the lifting (see **Lemma** 2.18 details).

Finally, suppose $F$ is a path homotopy. So $F$ is a path homotopy between $f := F|_{I\times 0}$ and $g := F_{I\times 1}$ such that

$$F(0 \times I) = f(0) = g(0) = b_0 \quad , \quad F(1 \times I) = f(1) = g(1)$$

Then $\tilde{F}$ will be a path homotopy between $\tilde{f} := \tilde{F}|_{I\times 0}$ and $\tilde{g} := \tilde{F}|_{I\times 1}$ provided: $\tilde{f}(0) = \tilde{g}(0)$, $\tilde{f}(1) = \tilde{g}(1)$ and

$$\tilde{F}(0 \times I) = \tilde{f}(0) \quad , \quad \tilde{F}(1 \times I) = \tilde{f}(1)$$

Let us show that these hold. We have $F(0 \times I) = b_0$, i.e $F$ carries the left edge to $b_0$. Since $\tilde{F}$ is a lifting of $F$, we must have

$$\tilde{F}(0 \times I) \subseteq p^{-1}(b_0)$$

Now since $\tilde{F}$ is continuous, $0 \times I$ is connected and $p^{-1}(b_0)$ is discrete, we must have that $\tilde{F}(0 \times I)$ is a singleton. Since $\tilde{F}(0,0) = e_0$, we have $\tilde{F}(0 \times I) = e_0$. Similarly, suppose $F(1 \times I) = b_1$. Then we can argue that

$$\tilde{F}(1 \times I) = e_1$$

where $p(e_1) = b_1$. Also, we have

$$\tilde{f}(0) = \tilde{F}(0,0) = e_0 = \tilde{F}(0,1) = \tilde{g}(0)$$
$$\tilde{f}(1) = \tilde{F}(1,0) = e_1 = \tilde{F}(1,1) = \tilde{g}(1)$$

■

**Theorem 2.20.** *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let $f, g$ be paths in $B$ from $b_0$ to $b_1$. Let $\tilde{f}$ and $\tilde{g}$ be their lifts to $E$ starting at $e_0$. If $f$ and $g$ are path homotopic, then $\tilde{f}$ and $\tilde{g}$ are path homotopic, and in particular $\tilde{f}(1) = \tilde{g}(1)$.*

*Proof.* The proof follows from the previous lemma. Let $F : I \times I \to B$ be a path homotopy between $f$ and $g$. Let $\tilde{F} : I \times I \to E$ be the unique lifting of $F$ such that $\tilde{F}(0,0) = e_0$. So,

$$F|_{I\times 0} = f \quad , \quad F|_{I\times 1} = g$$

So $\tilde{F}|_{I\times 0}$ is a lifting of $f$ such that $\tilde{F}(0,0) = e_0$. Since $\tilde{f}$ is the unique such lifting, we see that $\tilde{F}|_{I\times 0} = \tilde{f}$ and similarly $\tilde{F}|_{I\times 1} = \tilde{g}$. So $\tilde{F}$ is a path homotopy between $\tilde{f}$ and $\tilde{g}$. ■

**Definition 2.10.** Let $p : E \to B$ be a covering map and let $b_0 \in B$. Let $e_0 \in p^{-1}(b_0)$. For $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be the unique lifting to $E$ such that $\tilde{f}(0) = e_0$.

Define the set map $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$

$$[f] \mapsto \tilde{f}(1)$$

(note that $\tilde{f}(1) \in p^{-1}(b_0)$ because $f$ is a loop at $b_0$). $\phi$ is a well-defined map by **Theorem** 2.20, and it is called the *lifting correspondence*. It depends on the point $e_0$.

**Theorem 2.21.** *Let $p : E \to B$ be a covering map and let $p(e_0) = b_0$. Let $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ be the lifting correspondence.*

(1) *If $E$ is path connected, then $\phi$ is surjective.*
(2) *If $E$ is simply connected, then $\phi$ is bijective.*

*Proof.* To prove (1), let $e_1 \in p^{-1}(b_0)$. Let $\tilde{f} : I \to E$ be a path from $e_0$ to $e_1$. Then let $f := p \circ \tilde{f} : I \to B$. Then $\phi([f]) = \tilde{f}(1) = e_1$.

To prove (2), we only need to prove injectivity, as surjectivity is guaranteed by (1). So, let $[f], [g] \in \pi_1(B, b_0)$ be such that $\phi([f]) = \phi([g])$. If $\tilde{f}$ and $\tilde{g}$ are the lifts of $f, g$ respectively such that $\tilde{f}(0) = \tilde{g}(0) = e_0$, then $\tilde{f}(1) = \tilde{g}(1)$. Since $E$ is simply connected, $\tilde{f} \sim_p \tilde{g}$. Say $\tilde{F}$ is a path homotopy between $\tilde{f}$ and $\tilde{g}$. Then, $p \circ \tilde{F}$ is a path homotopy between $f$ and $g$. So, $[f] = [g]$ in $\pi_1(B, b_0)$. ■

**Theorem 2.22.** *$\pi_1(S^1) \cong \mathbb{Z}$.*

*Proof.* We work with the covering map $p : \mathbb{R} \to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. Let $b_0 = (1,0) \in S^1$, and let $e_0 = 0 \in \mathbb{R}$. Since $\mathbb{R}$ is simply connected, by **Theorem** 2.21 we have a bijection

$$\phi : \pi_1(S^1, b_0) \to p^{-1}(b_0) = \mathbb{Z}$$

We claim that $\phi$ is, in fact, a homomorphism.

Let $[f], [g] \in \pi_1(S^1, b_0)$; let $\tilde{f}$, $\tilde{g}$ be their liftings to $\mathbb{R}$ starting at $e_0 = 0$. Let $n = \tilde{f}(1)$, $m = \tilde{g}(1)$. So, we see that $\phi([f]) = n$ and $\phi([g]) = m$. Consider the path $\tilde{g}' : I \to \mathbb{R}$ defined by $\tilde{g}'(s) = \tilde{g}(s) + n$. Then $p \circ \tilde{g}' = p \circ \tilde{g}$. So, $\tilde{g}'$ is a lifting of $g$. Further, $\tilde{f}(1) = n = \tilde{g}'(0)$. So, we can apply the operation $*$ to $\tilde{f}$ and $\tilde{g}'$. So, we see that $\tilde{f} * \tilde{g}'$ is a path in $\mathbb{R}$ from 0 to $n + m$. Note that $\tilde{f} * \tilde{g}'$ is a (unique) lifting of $f * g$ starting at 0:

$$p \circ (\tilde{f} * \tilde{g}') = (p \circ \tilde{f}) * (p \circ \tilde{g}') = f * g$$

Hence, we have

$$\phi([f] * [g]) = (\tilde{f} * \tilde{g}')(1) = m + n = \phi([f]) + \phi([g])$$

and this completes the proof. ∎

**Remark 2.22.1.** $\pi_1(S^1, b_0)$ is generated by the loop $f : I \to S^1$ given by $f(s) = (\cos 2\pi s, \sin 2\pi s)$.

2.4. **Retractions and Fixed Points.** Let $X$ be a space and let $A \subseteq X$. A *retraction* of $X$ onto $A$ is a continuous map $r : X \to A$ such that $r|_A = \text{id}$. If such a map exists, we say that $A$ is a retract of $X$.

**Lemma 2.23.** *If $A \subseteq X$ is a retract of $X$, then the map $j_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective, where $a \in A$ and $j : A \to X$ is the inclusion map.*

*Proof.* By definition, we have the following:

$$A \xrightarrow{j} X \xrightarrow{r} A$$

such that $r \circ j = \text{id}_A$. Applying $*$ to the above diagram, we get the following:

$$\pi_1(A, a) \xrightarrow{j_*} \pi_1(X, a) \xrightarrow{r_*} \pi_1(A, a)$$

such that $r_* \circ j_* = \text{id}$. Hence, $j_*$ is injective. ∎

**Theorem 2.24** (**No Retraction Theorem**). *$S^1$ is not a retract of $B^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.*

*Proof.* Just apply **Lemma** 2.23: if $S$ were a retract of $B^2$, we would get an injective map from $\mathbb{Z} \to 1$, where 1 is the trivial group. ∎

**Exercise 2.2.** Show that $S^1$ is a retract of $\mathbb{R}^2 \setminus \{0\}$.

**Solution.** To be completed.

**Theorem 2.25.** *Let $h : S^1 \to X$ be a continuous map. Then the following are equivalent.*

(1) *$h$ is nullhomotopic, i.e $h$ is homotopic to the constant map $S^1 \to X$.*
(2) *$h$ extends to a continuous map $k : B^2 \to X$.*
(3) *$h_*$ is the trivial map of fundamental groups.*

*Proof.* Let us first prove (1) $\Longrightarrow$ (2). Let $H : S^1 \times I \to X$ be a homotopy between $h$ and a constant map. The goal will be to *contract* $S^1 \times I$ to $B^2$ to obtain the desired map $k : B^2 \to X$.

Consider the map $\pi : S^1 \times I \to B^2$ given by

$$\pi(x, t) = (1 - t)x$$

Clearly, $\pi$ is identity on $S^1 \times 0$. $\pi$ is constant on $S^1 \times 1$ and $\pi$ is injective on $S^1 \times t$ for all $t \neq 1$. It is also true that $\pi$ is a continuous, closed, surjective map (need to prove this!). Note that $H$ is constant on $S^1 \times I$ and $H = f$ on $S^1 \times 0$. We define the map $k : B^2 \to X$ as follows.

- $k(0) := H(S^1 \times 1) \in X$.
- If $x \in B^2 \setminus \{0\}$, $k(x) := H(\pi^{-1}(x)) \in X$: this is valid since $\pi|_{(S^1 \times I)\setminus(S^1 \times 1)} : (S^1 \times I) \setminus (S^1 \times 1) \to B^2 \setminus \{0\}$ is bijective (prove this!)

We claim that $k$ is the required extension. Let $x \in S^1 \subseteq B^2$. Then $k(x) = H(\pi^{-1}(x)) = H(x, 0) = h(x)$. Next, we show that $k$ is continuous. Let $A \subseteq X$ be a closed set. Then $k^{-1}(A) = \pi(H^{-1}(A))$. Then, $k^{-1}(A)$ is closed since $H$ is continuous and $\pi$ is closed.

Next, let us show that (2) $\implies$ (3). Let $S^1 \overset{j}{\hookrightarrow} B$ be the inclusion map. So, we have an extension of $h$

$$S^1 \overset{j}{\hookrightarrow} B^2 \overset{k}{\to} X$$

i.e $k \circ j = h$. This implies that

$$h_* = k_* \circ j_*$$

So, we have

$$\pi_1(S^1, b_0) \xrightarrow{j_*} \pi_1(B^2, b_0) \xrightarrow{k^*} \pi_1(X, h(b_0))$$

Since $B^2 \subseteq \mathbb{R}^2$ is convex, we have that $\pi_1(B^2, b_0) \cong 1$, where 1 is the trivial group. So, it follows that $h_*$ is the trivial map.

Finally, let us show that (3) $\implies$ (1). Let $p : \mathbb{R} \to S^1$ be the covering map $p(x) = (\cos 2\pi x, \sin 2\pi x)$. Denote by $p_0$ the restriction of $p$ to $I = [0, 1]$. Then, $[p_0]$ generated $\pi_1(S^1, b_0)$. Let $h(b_0) = x_0 \in X$; since $h_*$ is the trivial homomorphism, we see that $f := h \circ p_0$ represents the identity element of $\pi_1(X, x_0)$. Let $F$ be a path homotopy between $f$ and the constant loop at $x_0$.

Now, consider the map $\tilde{p_0} := p_0 \times \mathrm{id} : I \times I \to S^1 \times I$. This map is a continuous, closed, surjective map (prove this). We have the following.

$$\tilde{p_0}(0 \times t) = b_0 \times t \qquad \forall t \in I$$
$$\tilde{p_0}(1 \times t) = b_0 \times t \qquad \forall t \in I$$
$$\tilde{p_0} \text{ is injective outside } 0 \times I, 1 \times I$$

Since $F$ is a path homotopy, we have

$$F(0 \times I) = x_0 = F(1 \times I) = F(I \times 0)$$

and that $F|_{I\times 1} = f = h \circ p_0$. We can now define a map $H : S^1 \times I \to X$ as follows: $H(x, 0) = x_0$ for all $x \in S^1$, $H(x, 1) = h(x)$ for all $x \in S^1$ and if $(x, t) \in S^1 \times I$ for $t \neq 0, 1$, we define $H(x, t) = F((p_0 \times \mathrm{id})^{-1}(x, t)) \in X$. Check that $H$ is continuous, and $H$ is a homotopy between $h$ and the constant map $x \to x_0$. ∎

**Corollary 2.25.1.** *The following are true.*

1. *The inclusion $j : S^1 \to \mathbb{R}^2 \setminus \{0\}$ is not null-homotopic.*
2. *The identity $i : S^1 \to S^1$ is not null-homotopic.*

*Proof.* We first show (1). By **Exercise** 2.2, we know that $S^1$ is a retract of $\mathbb{R}^2 \setminus \{0\}$: a retraction is given by $r : \mathbb{R}^2 \setminus \{0\} \to S^1$

$$x \mapsto \frac{x}{||x||}$$

So $j_* : \pi_1(S^1, b_0) \to \pi_1(\mathbb{R}^2 \setminus \{0\}, b_0)$ is injective. So $j$ cannot be null-homotopic by **Theorem** 2.25.

(2) is straightforward by **Theorem** 2.25. ■

2.5. **Brouwer and Borsuk-Ulam.** In this section, we will use these tools to prove some useful facts.

**Theorem 2.26** (**Brouwer's Fixed Point Theorem**). *If $f : B^2 \to B^2$ is a continuous map, then there is a point $x \in B^2$ such that $f(x) = x$.*

*Proof.* Suppose $f(x) \neq x$ for all $x \in B^2$. Consider the ray from $f(x)$ passing through $x$, and suppose this ray meets $S^1$ at $r(x)$. Then we have a map $r : B^2 \to S^1$ given by $x \to r(x)$. Then

(1) $r$ is well-defined, since $x \neq f(x)$ for all $x \in B$.
(2) $r$ is continuous, since $f$ is.
(3) If $x \in S^1$, then $r(x) = x$.

So $r : B^2 \to S^1$ is a retraction. But we know that there cannot be such a retraction, and hence this is a contradiction. So, $f$ must have a fixed point. ■

**Theorem 2.27** (**Borsuk-Ulam**). *Any continuous map $f : S^2 \to \mathbb{R}^2$ satisfies $f(x) = f(-x)$ for some point $x \in S^2$.*

*Proof.* Suppose the theorem is not true. Consider the map $g : S^2 \to S^1$ given by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

Clearly, $g$ is continuous by hypothesis. Let $\eta : I \to S^2$ be the function $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$. Note that $\eta$ is a loop which circles the equator of $S^2$ once. Let $h = g \circ \eta$.

Now, observe that $g(-x) = -g(x)$ for all $x \in S^2$. Also, note that

$$\begin{aligned} h\left(s + \frac{1}{2}\right) &= g(\cos(2\pi s + \pi), \sin(2\pi s + \pi), 0) \\ &= g(-\cos(2\pi s), -\sin(2\pi s), 0) \\ &= -g(\cos(2\pi s), \sin(2\pi s), 0) \\ &= -h(s) \end{aligned}$$

As usual, let $p : \mathbb{R} \to S^1$ be the covering map $t \mapsto (\cos(2\pi t), \sin(2\pi t))$. Let $\tilde{h}$ be a lifting of $h$ to a path in $\mathbb{R}$. Then

$$h\left(s + \frac{1}{2}\right) = -h(s) \implies \tilde{h}\left(s + \frac{1}{2}\right) = \tilde{h}(s) + \frac{q}{2}$$

for some odd integer $q$. We note that $q$ depends continuously on $s$, but since $q$ is an integer, it must be a constant. So,

$$\tilde{h}(1) = \tilde{h}(1/2) + q/2 = \tilde{h}(0) + q/2 + q/2 = \tilde{h}(0) + q$$

Hence $[h] = q \cdot$ a generator of $\pi_1(S^1, h(0))$. So $[h]$ is not trivial.

Consider the map $g_* : \pi_1(S^2, \eta(0)) \to \pi_1(S^1, h(0))$ which satisfies $[\eta] \mapsto [h] \neq [e]$. But observe that $[\eta]$ is the trivial loop: this follows because the upper hemisphere of $S^2$ is homeomorphic to $B^2$ and hence $\eta$ is null-homotopic. This is a contradiction. ■

2.6. **Fundamental Theorem of Algebra.** In this section, we will prove the fundamental theorem of algebra, which says that every non-constant polynomial $f(x) \in \mathbb{C}[x]$ has a root in $\mathbb{C}$. We now prove this.

Without loss of generality, suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is monic. Suppose that $f(x)$ has no complex roots. Consider the map $\mathbb{C} \to S^1$ given by

$$z \mapsto \frac{f(z)}{|f(z)|}$$

We choose an integer $r$ such that $r > \max\{|a_0| + ... + |a_{n-1}|, 1\}$. Let $S^1_r = \{z \in \mathbb{C} \mid |z| = r\}$. Then we have the map

$$S^1_r \hookrightarrow \mathbb{C} \to S^1$$

given by

$$\varphi(z) = \frac{f(z)}{|f(z)|}$$

Let $\eta : I \to S^1_r$ be the loop at $r$ given by $\eta(s) = re^{2\pi i s}$. Then we claim that $\varphi_*([\eta])$ is equal to $n$ times the generator of $\pi_1(S^1, \varphi(r))$, where $n$ as above is the degree of $f$. To show this, note that we have maps $I \xrightarrow{\eta} S^1_r \xrightarrow{\varphi} S^1$ which are

$$s \mapsto re^{2\pi i s} \mapsto \frac{f(re^{2\pi i s})}{|f(re^{2\pi i s})|}$$

Now, the generator of $\pi_1(S^1, \varphi(r))$ is the following loop at $\varphi(r)$:

$$\rho : I \to S^1, \qquad s \mapsto \varphi(r)e^{2\pi i s} = \frac{f(r)}{|f(r)|}e^{2\pi i s}$$

Now, $n\rho : I \to S^1$ is the loop

$$s \mapsto \frac{f(r)}{|f(r)|}e^{2\pi i n s}$$

We show that $[n\rho]$ and $\varphi_*(\eta)$ are equal.

For $0 \le t \le 1$, set $f_t(x) = x^n + t(a_{n-1}x^{n-1} + ... + a_1x + a_0)$. Then for $|z| = r$ and $0 \le t \le 1$ we have

$$\begin{aligned} |z|^n = |z| \cdot |z|^{n-1} = r|z|^{n-1}(|a_{n-1}| + \cdots + |a_0||)|z^{n-1}| \\ \ge |a_{n-1}z^{n-1}| + \cdots + |a_1z| + |a_0| \\ \ge |a_{n-1}z^{n-1} + \cdots + a_1z + a_0| \end{aligned}$$

because $r \ge 1$. This shows that $f_t(z) \ne 0$ for $0 \le t \le 1$ and $|z| = r$.

Next, define $F : I \times I \to S^1$ by

$$F(s,t) = \frac{f(r)}{|f(r)|}\frac{f_t(re^{2\pi i s})/f_t(r)}{|f_t(re^{2\pi i s})/f_t(r)|}$$

Since $f_t(z) \ne 0$ for all $0 \le t \le 1$ and $|z| = r$, we see that $F(s,t)$ is well-defined.

Now we show that $F$ is a path homotopy between $n\rho$ and $\varphi \circ \eta$, which will prove our claim.

(1) Clearly, $F$ is a continuous function.

(2) Suppose $t = 0$. So we have that $f_0(x) = x^n$. So,

$$\begin{aligned} F(s, 0) &= \frac{f(r)}{|f(r)|} \frac{f_0(re^{2\pi is})/f_0(r)}{|f_0(re^{2\pi is})/f_0(r)|} \\ &= \frac{f(r)}{|f(r)|} e^{2\pi ins} \\ &= n\rho \end{aligned}$$

(3) Suppose $t = 1$. Note that $f_1 = f$. So,

$$\begin{aligned} F(s, 1) &= \frac{f(r)}{|f(r)|} \frac{f_1(re^{2\pi is})/f_1(r)}{|f_1(re^{2\pi is})/f_1(r)|} \\ &= \frac{f(re^{2\pi is})}{|f(re^{2\pi is})|} \\ &= \varphi \circ \eta \end{aligned}$$

(4) Finally, suppose $s = 0, 1$. Then,

$$F(0, t) = F(1, t) = \frac{f(r)}{|f(r)|} \frac{f_t(r)/f_t(r)}{|f_t(r)/f_t(r)|} = \frac{f(r)}{|f(r)|} = \varphi(r)$$

and this proves the claim, i.e this proves that $[n\rho] = \varphi_*(\eta)$. Since $[\eta] \in \pi_1(S^1_r, r)$ is a generator, we conclude that $\varphi_*$ is non-trivial homomorphism. However, note that

$$\pi_1(S^1_r, r) \xrightarrow{\varphi_*} \pi_1(S^1, \varphi(r))$$

given by the composition

$$S^1_r \hookrightarrow \mathbb{C} \to S^1$$

is trivial, because $\pi_1(\mathbb{C})$ is the trivial group. This is a contradiction.

### 2.7. Some nice exercises.

Here are some nice exercises to try.

**Exercise 2.3.** Is $S^1$ a retract of $\mathbb{R}$?

**Solution.** No, because $\pi_1(\mathbb{R})$ is trivial.

**Exercise 2.4.** Let $f : B^2 \to B^2$ be a continuous function such that $f(x) = x$ for all $x \in S^1$. Show that $f$ is surjective. (Hint: Use the idea in the proof of the Brouwer fixed point theorem).

**Exercise 2.5.** Let $f : S^1 \to \mathbb{R}$ be a continuous function. Show that there exists $x \in S^1$ such that $f(x) = f(-x)$ (Hint: use the intermediate value theorem).

**Solution.** Solution on this link.

**Exercise 2.6.** A space $X$ is simply connected if and only if all continuous functions $S^1 \to X$ are null-homotopic.

**Exercise 2.7.** Let $p : E \to B$ be a covering map, with $E$ path connected and $B$ simply connected. Show that $p$ is a homeomorphism.

2.8. **Deformation Retracts.** In this section, we will see some special cases where computing fundamental groups is nicer.

**Definition 2.11.** Let $A$ be a subspace of a space $X$. We say that $A$ is a *deformation retract* of $X$ if the following hold.

(1) $A$ is a retract of $X$, i.e there exists a retraction $r : X \to A$.
(2) $j \circ r$ and the identity map $i : X \to X$ are homotopic (where $J$ is the inclusion of $A$ in $X$). In other words, there is some $H : X \times I \to X$ such that $H(x, 0) = x$, $H(x, 1) \in A$, $H(a, 1) = a$ for all $x \in X$, $a \in A$.

In this case, $H$ is called a *deformation retraction*.

**Example 2.4.** Let $n \geq 1$. We show that $S^n$ is a deformation retract of $\mathbb{R}^{n+1} - \{0\}$. First, note that $S^n$ is a retract of $\mathbb{R}^{n+1} - \{0\} = X$:

$$r : X \to S^n, \qquad r(x) = \frac{x}{||x||}$$

The required homotopy is given by

$$H : X \times I \to X, \qquad H(x, t) = (1-t)x + t\frac{x}{||x||}, \qquad t \in I, x \in I$$

It is easy to check that $H$ has the required properties.

**Theorem 2.28.** *Let $A$ be a deformation retract of $X$. Let $x_0 \in A$. Then the inclusion map $j : A \hookrightarrow X$ induces an isomorphism of fundamental groups $j_* : \pi_1(A, x_0) \xrightarrow{\sim} \pi_1(X, x_0)$.*

*Proof.* Let $r : X \to A$ be a retraction. Since $j \circ r : X \to X$ is homotopic to the identity map $i : X \to X$, we know that $(j \circ r)_* = i_*$ (prove this), and hence $(j \circ r)_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identity map. On the other hand, $r \circ j = i_A : A \to A$ (where $i_A$ is the identity map). So $r_* \circ j_*$ is the identity map on $\pi_1(A, x_0)$. So, $j_*$ is an isomorphism. ■

**Corollary 2.28.1.** *We have $\pi_1(S^n, x_0) \cong \pi_1(\mathbb{R}^{n+1} - 0, x_0)$, where $x_0 \in S^n$. In particular, $\pi_1(\mathbb{R}^2 - 0, x_0) \cong \mathbb{Z}$.*

**Exercise 2.8.** Let $X, Y$ be spaces and let $x_0, y_0 \in X, Y$. Show that

$$\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

**Solution.** Full solution needs to be written, but here is the idea. Let $p, q$ be the two projections $X \times Y \to X$ and $X \times Y \to Y$. Consider $\phi : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ given by

$$[f] \mapsto p_*([f]) \times q_*([f])$$

Show that $\phi$ is an isomorphism.

**Exercise 2.9.** Let $x \in S^1$. Show that $S^1 \times x_0$ is a retract of $S^1 \times S^1$, but it is not a deformation retract.

2.9. **A special form of Van Kampen's Theorem.** We will now look at a special version of a very important theorem.

**Theorem 2.29** (**Van Kampen**). *Suppose $X = U \cup V$, where $U, V$ are open and $U \cap V$ is path connected. Let $x_0 \in U \cap V$. Let $i : U \hookrightarrow X$ and $j : V \hookrightarrow X$ be the inclusions. Then the images of the induced homomorphisms*

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$$
$$j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

*generate $\pi_1(X, x_0)$.*

*Proof.* Let $[f] \in \pi_1(X, x_0)$. We show that $f \sim_p g_1 * \cdots * g_n$, where each $g_i$ is a loop at $x_0$ that lies in either $U$ or $V$. This will complete the proof of the claim.

**Step 1.** *There exists a subdivision $a_0 = 0 < a_1 < \cdots < a_n = 1$ of $I$ such that $f(a_i) \in U \cap V$ and $f([a_i, a_{i+1}])$ is contained in $U$ or in $V$ for all $i$.*

To prove this, we will use the **Lebesgue Number Lemma** 1.30. We consider the open cover $f^{-1}(U)$ and $f^{-1}(V)$ of $I$. Applying the lemma, we can find a subdivision $b_0 < \cdots < b_m$ of $I$ such that $f([b_{i-1}, b_i]) \subseteq U$ or $f([b_{i-1}, b_i]) \subseteq V$.

Now if each $f(b_i) \in U \cap V$, we are done. Suppose not, say $f(b_i) \notin U \cap V$. We know $f([b_{i-1}, b_i]) \subseteq U$ or $V$ and $f([b_i, b_{i+1}]) \subseteq U$ or $V$. So,

$$f(b_i) \in U \implies f([b_{i-1}, b_{i+1}]) \subseteq U \qquad (\text{since } f(b_i) \notin V \text{ in this case})$$
$$f(b_i) \in V \implies f([b_{i-1}, b_{i+1}]) \subseteq V \qquad (\text{since } f(b_i) \notin U \text{ in this case})$$

So we may delete $b_i$ from the subdivision. Continuing this way, we obtain the required subdivision.

**Step 1.** Let $a_0 < a_1 < \cdots a_n$ be the subdivision in **Step 1**. Let $f_i$ be the path in $X$ that equals the positive linear map of $[0, 1]$ onto $[a_{i-1}, a_i]$ followed by $f$. By the proof of part (1) of **Theorem** 2.6 (the proof of associativity of the group operation), we see that

$$f \sim_p f_1 * f_2 * \cdots * f_n$$

But note that $f_i$ are not necessarily loops in $X$. We get loops as follows: note that $x_0$ and $f(a_i) = f_i(1) = f_{i+1}(0)$ are both in $U \cap V$. Since $U \cap V$ is path connected, let $\alpha_i$ be a path in $U \cap V$ from $x_0$ to $f(a_i)$. Let $\alpha_0$, $\alpha_n$ be the constant paths at $x_0$. Now define

$$g_i := \alpha_{i-1} * f_i * \overline{\alpha_i}$$

Then $g_i(0) = \alpha_{i-1}(0) = x_0$ and $g_i(1) = \overline{\alpha_i}(1) = x_0$. So each $g_i$ is a loop at $x_0$, and moreover, the image of $g_i$ lies in either $U$ or $V$. We have

$$f \sim_p f_1 * \cdots * f_n \sim_p g_1 * \cdots * g_n$$

and hence the proof is complete. ■

**Corollary 2.29.1.** *If $X = U \cup V$, $U, V \subseteq X$ are open and $U \cap V$ is path connected and $U, V$ are simply connected, then $X$ is simply connected.*

**Theorem 2.30.** *$S^n$ is simply connected for $n \geq 2$.*

*Proof.* Let $p = (0, 0, ..., 1) \in S^n$ and $q = (0, 0, ..., -1) \in S^n$ be fixed. Via the stereographic projections, we know that

$$S^n - p \cong S^n - q \cong \mathbb{R}^n$$

Now let $U = S^n - p$ and $V = S^n - q$. Then $U, V$ are open in $S^n$ and $U \cup V = S^n$. Clearly, $U, V$ are simply connected. Also,

$$U \cap V \cong S^n - p - q \cong \mathbb{R}^n - \text{ a point}$$

Since $n \geq 2$, we see that $\mathbb{R}^n -$ a point is path connected. So $S^n$ is simply connected by the previous theorem. ■

**Corollary 2.30.1.** *Since $S^n$ is a deformation retract of $\mathbb{R}^n - 0$, it follows that $\mathbb{R}^n - 0$ is simply connected if $n \geq 3$.*

**Corollary 2.30.2.** *If $\mathbb{R}^2 \cong \mathbb{R}^n$ then $n = 2$.*

*Proof.* We already know that $\mathbb{R} \not\cong \mathbb{R}^2$. So, let $n \geq 3$. If there is a homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^n$, then we get a homeomorphism $\varphi' : \mathbb{R}^2 - 0 \to \mathbb{R}^n - \varphi(0)$. So, in that case, we will get

$$\mathbb{Z} \cong \pi_1(\mathbb{R}^2 - 0, x) \cong \pi_1(\mathbb{R}^n - \varphi(0), \varphi(x)) \cong 1$$

which is not possible. ■