On the Critical Value of Sandpiles on the Infinite Ladder Graph

Ashwin Padaki and Siddhant Chaudhary

Program in Mathematics for Young Scientists (PROMYS)

Abstract

Mod-1 harmonic functions can be used to show nontrivial upper bounds on critical density for sandpiles on infinite ladder graphs.

Introduction

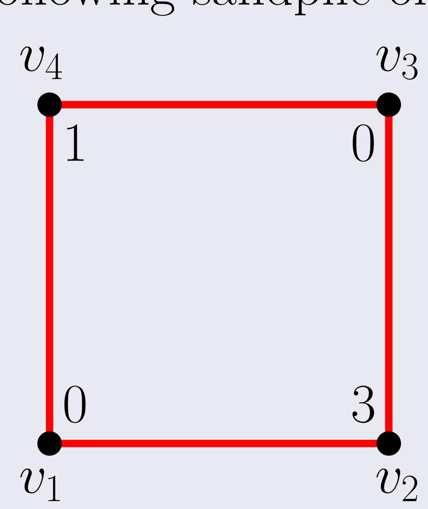
A **sandpile** consists of a graph and a function $s : V \to \mathbb{Z}$ which determines the number of chips at each vertex. We call a vertex v **stable** if:

$$s(v) < \deg(v).$$

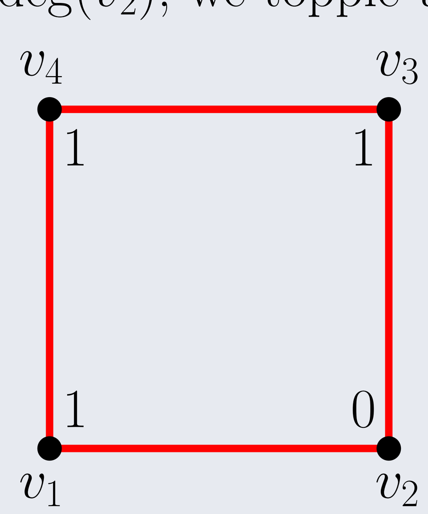
Otherwise, it is **unstable** and must be toppled. The process of toppling consists of removing $\deg(v)$ chips from vertex v, and adding 1 chip to each of its neighboring vertices. If there does not exist a sequence of topplings that stabilizes an entire sandpile, we say the sandpile **explodes**.

Example

Consider the following sandpile on C_4:

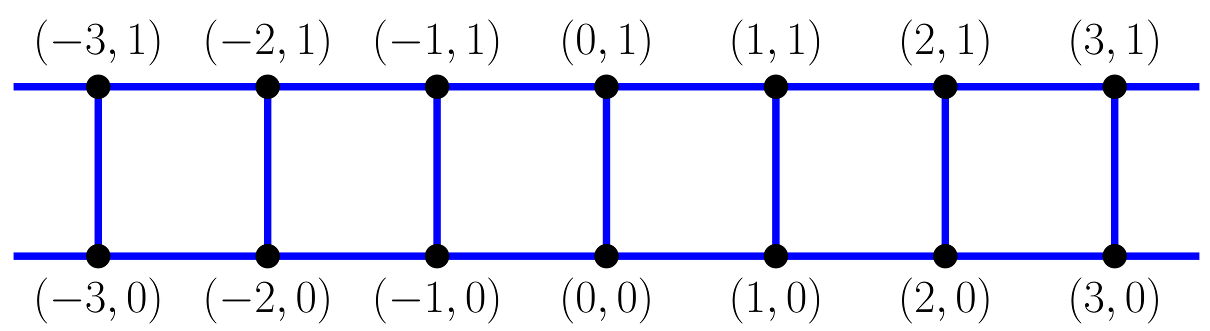


Since $s(v_2) \geq \deg(v_2)$, we topple v_2 and get:



All vertices are now stable so we call the sandpile **stabilizable**.

Introduction (cont'd)



(the infinite ladder graph)

A Bernoulli sandpile is one that is initialized as:

$$s(v) = \begin{cases} \deg(v) & \text{with probability } p \\ 0 & \text{with probability } (1-p) \end{cases}$$

Note that all vertices in $\mathbb{Z} \times \mathbb{Z}_2$ have degree 3.

Main Problem

We aim to find p_τ, the lowest value such that for all $p \geq p_\tau$:

$$\mathbb{P}\left(s \text{ explodes} = 1\right)$$

where s is a Bernoulli sandpile with parameter p.

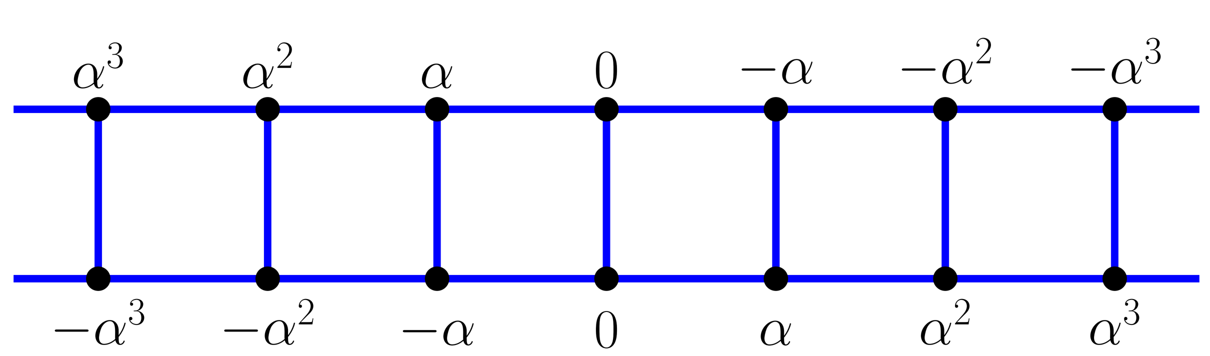
To approach the problem we use the theory of mod-1 harmonic functions. A function $h : V \to \mathbb{R}$ is said to be mod-1 harmonic if for all $v \in V$:

$$\sum_{w \sim v} h(w) = \deg(v) h(v).$$

We want h to have three useful properties:

- Horizontal antisymmetry: $h(x, y) = -h(-x, y)$
- Vertical antisymmetry: $h(x, y) = -h(x, -y)$
- $\lim_{n \to \infty} h(n, 0) = 0$

We construct a mod-1 harmonic function h that satisfies these properties on $\mathbb{Z} \times \mathbb{Z}_2$. Using $\alpha = 2 - \sqrt{3}$, h takes values:



Results

Theorem

On the infinite ladder graph $\mathbb{Z} \times \mathbb{Z}_2$:

$$p_\tau \leq 2/3 - c \text{ where } c \approx 0.03.$$

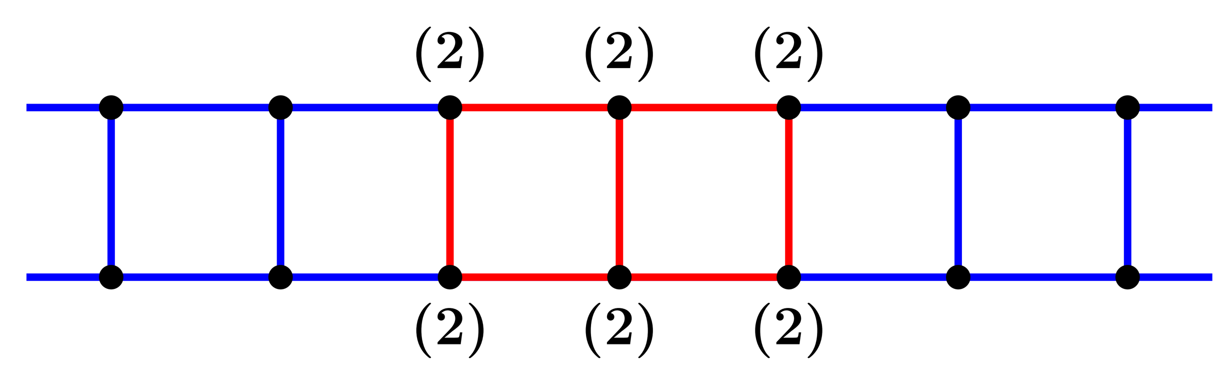
On $\mathbb{Z} \times \mathbb{Z}_2$, let s be a stabilizable Bernoulli sandpile with parameter $p = 2/3 - \varepsilon$. Let $s_0, s_1, s_2, \ldots, s_\infty$ be a sequence of sandpiles generated by a sequence of topplings, where s_∞ is our final state. Using our mod-1 harmonic function h, we define (s_k, h) as:

$$(s_k, h) = \sum_{v \in V} s_k(v) h(v).$$

The exponential decay of h ensures that (s_0, h) and (s_∞, h) converge and mod-1 harmonicity of h ensures that $(s_0, h) \equiv (s_\infty, h) \mod 1$.

We treat $(s_0, h) \mod 1$ and $(s_\infty, h) \mod 1$ as equivalent random variables which, then, must have **equal variance**. Since (s_0, h) is a sum of independent Bernoulli trials, we can show that:

$$\mathrm{Var}\left[(s_0, h) \mod 1\right] = \frac{36\alpha^2}{1 - \alpha^2} \cdot p(1-p).$$



Next, using Conservation of Density [2] and a union bound, we can show that:

$$\mathbb{P}\big(s_\infty(v) = 2 \mid v \in [-1, 1] \times [0, 1]\big) \geq 1 - 18\varepsilon.$$

We use this result to find a lower bound on $\mathrm{Var}\left[(s_\infty, h) \mod 1\right]$. Setting the two variance expressions equal and solving for ε, we find that $\varepsilon \geq c$ where $c \approx 0.03$. Then, for all $p \geq 2/3 - c$ any sandpile with parameter p will explode. The theorem follows by the definition of p_τ.

Extension to $\mathbb{Z} \times \mathbb{Z}_m$

For all $m \geq 2$, we are able to construct a mod-1 harmonic function h on the extended ladder graph $\mathbb{Z} \times \mathbb{Z}_m$. We take $h(v) = 0$ at each vertex v except for those in a single $\mathbb{Z} \times \mathbb{Z}_2$ subgraph, in which h is constructed the same way as before. Instead of $\alpha = 2 - \sqrt{3}$, we find a class of constants:

$$\alpha(m) = \frac{(m+2) - \sqrt{(m+2)^2 - 4}}{2}.$$

Through similar variance bounds, we are able to achieve analogous results.

Conclusion

Our approach of treating $(s_0, h) \mod 1$ and $(s_\infty, h) \mod 1$ as random variables not only yields bounds on this critical value for Bernoulli sandpiles on the infinite ladder graph, but extends nicely to other infinite graphs and probability distributions as well.

Future Work

- Achieve a stronger bound on the variance of (s_∞, h) and hence p_τ.
- Prove that $p_\tau \geq 1/2$ and tighten this bound.

References

[1] [CP18] Corry, S., and Perkinson, D. Divisors and Sandpiles, American Mathematical Society. 2018.

[2] [LMPU16] Levine, L., Murugan, M., Peres, Y., and Ugurcan, B. The divisible sandpile at critical density. Annales Henri Poincare (2017) 17 (7) : 1677-1711

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