Bernoulli Sandpiles on the Infinite Ladder Graph

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Presentation Outline

1. Basic Definitions and Properties of Sandpiles
2. Bernoulli Sandpiles on the Ladder Graph
3. *Mod-1 Harmonic* Functions
4. Upper Bound for p_τ

Sandpiles and Relevant Definitions

Definition

A *sandpile* (V, E, s) consists of an undirected graph and a function $s : V \to \mathbb{Z}$.

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The function $s(v)$ identifies each vertex $v \in V$ with a number of "chips".

To initialize a sandpile, a certain number of chips is placed on each vertex.

Sandpiles and Relevant Definitions

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$$s(v) < \deg(v).$$

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Otherwise, it is *unstable* and must be toppled.

Toppling a vertex v involves removing $\deg(v)$ chips from v, and adding 1 chip to each of its neighboring vertices (this is repeated until v stabilizes).

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We say that a sandpile is *stabilizable* if there exists a sequence of topplings that will result in a stable sandpile.

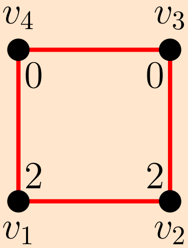
We say a sandpile *explodes* or is *unstabilizable* if no such sequence of topplings exists.

Examples of Sandpiles

We consider a stabilizable sandpile on C_4

Example 1

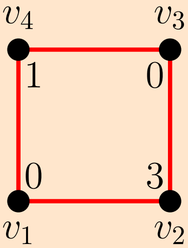
Notice that $2 = s(v_1) \geq \deg(v_1)$.



Examples of Sandpiles

Example 1

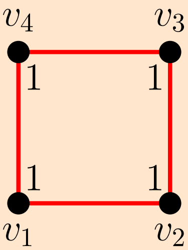
(after toppling v_1)



Examples of Sandpiles

Example 1

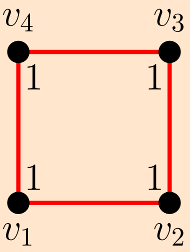
(after toppling v_2)



Examples of Sandpiles

Example 1

(after toppling v_2)



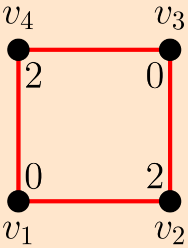
All vertices have fewer than 2 chips $\implies$ stable

Examples of Sandpiles

We consider an unstabilizable sandpile on C_4.

Example 2

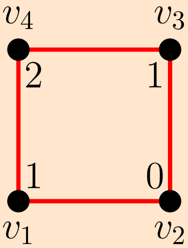
Notice that $2 = s(v_2) \geq \deg(v_2)$



Examples of Sandpiles

Example 2

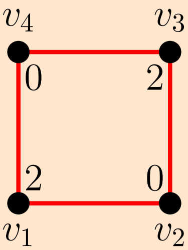
(after v_2 topples)



Examples of Sandpiles

Example 2

(after v_4 topples)



Topplings repeat indefinitely $\implies$ explodes.

Properties of Sandpiles

Definition

The *odometer* $u(v)$ of a vertex v is the number of times that v topples.

Definition

A toppling of a vertex v is *illegal* if $s(v) < \deg(v)$. A sequence of topplings is *illegal* if it contains at least one illegal toppling

We can now state 2 interesting properties of sandpiles.

Properties of Sandpiles

Remark

Least Action Principle: Let T_1 and T_2 both be sequences of topplings that stabilize a sandpile s. If T_1 is legal and T_2 is not necessarily legal then

$$|T_1| \leq |T_2| .$$

Properties of Sandpiles

Remark

Abelian Property: Given a sandpile, any permutation of a sequence of topplings will yield the same ending configuration (including illegal ones).

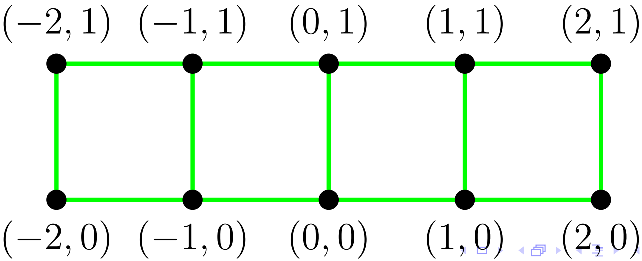
Ladder Graph

We focus on sandpiles on the ladder graph.

Definition

The *finite ladder graph* is $L_n = [-n, n] \times \mathbb{Z}_2$ where adjacent vertices are connected

The ladder graph L_2:



Ladder Graph

Definition

The *infinite ladder graph* $L = \mathbb{Z} \times \mathbb{Z}_2$ is an infinite extension of the ladder graph.

In our project, we work on the infinite ladder graph. Note that in the infinite ladder graph, each vertex has a degree of 3.

Bernoulli Sandpiles

Definition

For a given graph L_n, we define a *Bernoulli Sandpile* on L_n as the sandpile initialized by:

$$s(v) = \begin{cases} 3 & \text{with probability } p \\ 0 & \text{with probability } (1-p) \end{cases}$$

This can be extended to the infinite ladder L as well.

Bernoulli Sandpiles

Definition

We define

$$p_c(n) := \text{inf}\left\{p : \mathbb{P}_{p,n}\left(s \text{ explodes} \geq \frac{1}{2}\right)\right\}.$$

The number $p_c(n)$ is called the *critical* or *threshold* value of p for the Bernoulli sandpile.

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$$p_c(n) := \inf \left\{ p : \mathbb{P}_{p,n} \left(s \text{ explodes} \geq \frac{1}{2} \right) \right\}.$$

The number $p_c(n)$ is called the *critical* or *threshold* value of p for the Bernoulli sandpile. We can extend this to L and define

$$p_\tau := \inf \left\{ p : \mathbb{P}_{p} \left(s \text{ explodes} \geq \frac{1}{2} \right) \right\}.$$

Numerical Data for p_c

Using a Java program, we approximated $p_c(n)$ for some values of n.

n	$p_c(n)$
5	0.532
10	0.542
15	0.545
20	0.547
25	0.548
30	0.549
35	0.550
40	0.550
45	0.551
50	0.551

Conservation of Density

Definition

We define $\mathbb{E}s(v)$ as the expected number of chips at a vertex v in a sandpile s.

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Proposition

s_∞ is uniquely determined by s.

Conservation of Density

In our proof we make use of an important theorem:

Theorem

Let s be a Bernoulli sandpile on L_n or L.

Then

$$\mathbb{E}s(v) = \mathbb{E}s_\infty(v)$$

for all vertices v.

Harmonic Functions (mod 1)

Definition

For a function $h : V \to \mathbb{R}$, we define the *Laplacian* Δh to be the function

$$\Delta h(v) := \left(\sum_{w \sim v} h(w) \right) - \text{deg}(v) h(v)$$

Harmonic Functions (mod 1)

Definition

A function $h : V \to \mathbb{R}$ is said to be *harmonic* if it satisfies $\Delta h(v) = 0$ for all v.

Definition

A function $h : V \to \mathbb{R}$ is said to be *mod-1-harmonic* if it satisfies $\Delta h(v) \in \mathbb{Z}$ for all v.

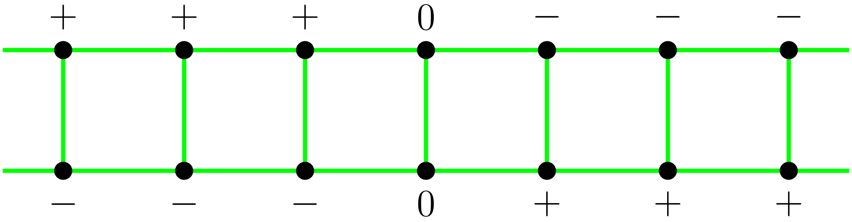
Harmonic Functions (mod 1)

We constructed a harmonic function on L:

$$h(0,0) = 0$$

$$h(n,0) = \left\{ B_n(2-\sqrt{3}) \right\} \quad (\forall n \in \mathbb{N})$$

$$h(x,y) = -h(x,1-y) \text{ and } h(x,y) = -h(-x,y)$$



Harmonic Functions (mod 1)

$$h(n,0) = \left\{ B_n(2-\sqrt{3}) \right\} \quad \forall n \in \mathbb{N}$$

The sequence B_n is defined recursively as:

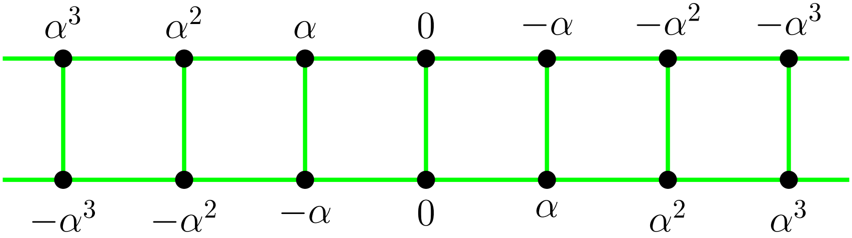
$$B_0 = 0$$
$$B_1 = 1$$
$$B_n = 4B_{n-1} - B_{n-2} \text{ (for } n \geq 2)$$

Harmonic Functions (mod 1)

We can prove that the other forms for h are:

$$h(0,0) = 0$$
$$h(n,0) = B_n \cdot \alpha - B_{n-1} = \alpha^n$$

for all $n \geq 1$, where $\alpha = 2 - \sqrt{3} = 0.2679\ldots$.



Harmonic Functions (mod 1)

Proposition

If h is *mod-1 harmonic* at a point $(x, y) \in L$, then it is also mod-1-harmonic at points $(x, 1 - y)$, $(-x, y)$ and $(-x, 1 - y)$.

Corollary

h is *mod-1 harmonic* at every vertex $(x, y) \in L$.

Upper Bound for p_τ

Definition

Let s be a sandpile on L. For the function h, define

$$(s, h) := \sum_{v \in V} h(v)s(v)$$

Upper Bound for p_τ

Lemma

Let s be a stabilizable Bernoulli sandpile on L. Then (s, h) and (s_∞, h) are well-defined, and:

$$(s, h) \equiv (s_\infty, h) \pmod 1$$

or otherwise stated, $(s_\infty, h) - (s, h) \in \mathbb{Z}$

Proof Outline

Induction on sandpile states $(s_0, s_1, s_2, \ldots, s_\infty)$

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Recall:

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Induction on sandpile states $(s_0, s_1, s_2, \ldots, s_\infty)$

Recall:

$$\Delta h(v) := \left(\sum_{w \sim v} h(w) \right) - \deg(v) h(v).$$

Therefore: $(s_{i+1}, h) - (s_i, h) = \Delta h(v) \in \mathbb{Z}$.

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Induction on sandpile states $(s_0, s_1, s_2, \ldots, s_\infty)$

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Therefore: $(s_{i+1}, h) - (s_i, h) = \Delta h(v) \in \mathbb{Z}$.

Also:

$$|(s,h)|, |(s_\infty, h)| \leq 6 \sum_{n=0} \alpha^n$$

Upper Bound for p_τ

Theorem

Let s be a Bernoulli sandpile on L with $p = \frac{2}{3}$. Then,

$$\mathbb{P}(s \text{ explodes}) = 1.$$

Corollary

$$p_\tau < \frac{2}{3}.$$

Proof Outline

- $p = \frac{2}{3} \implies \mathbb{E}s(v) = 2$

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- $\forall v \in V, \mathbb{E}s_\infty(v) = 2 \implies s_\infty(v) = 2$
- This means that

$$(s_\infty, h) = 2 \sum_{v \in V} h(v) = 0$$

Proof Outline

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- $(s,h) \in \{-2,-1,0,1,2\}$

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Definition

For the purposes of our research, we call an event *translation invariant* if its occurrence is not impacted by translation.

Proof Outline

Then

$$(s, h) = 3 \sum_{\substack{h(v)>0 \\ s(v)\neq 0}} h(v) + 3 \sum_{\substack{h(v)<0 \\ s(v)\neq 0}} h(v) = k$$

Which gives us

$$\sum_{\substack{h(v)>0 \\ s(v)\neq 0}} h(v) + \sum_{\substack{h(v)<0 \\ s(v)\neq 0}} h(v) = \frac{k}{3}$$

(where $k \in \{0, \pm 1, \pm 2\}$)

Proof Outline

We will show an outline for the case $k = 0$, as the other cases require similar techniques.

- Case $k = 0$

$$(s, h) = \sum_{h(v)>0} h(v) + \sum_{h(v)<0} h(v) = 0$$

I

Proof Outline

which means that

$$\sum_{h(v)>0} h(v) = \sum_{h(v)<0} -h(v)$$

We can express the left and right hand sides as:

$$c_1\alpha^{a_1} + c_2\alpha^{a_2}... = d_1\alpha^{b_1} + d_2\alpha^{b_2}...$$

where $a_i, b_i \in \mathbb{N}$ and $c_i, d_i \in \{1, 2\}$

Further Conjectures

- $\lim_{n\to\infty} p_c(n) = p_\tau$.
- If s is a Bernoulli sandpile on L (or L_n) such that $\mathbb{E}s(v) < 1.5$, then $\mathbb{P}(s \text{ stabilizes}) = 1$. This gives a lower bound on p_τ, which is

$$p_\tau \geq \frac{1}{2}.$$

- The simulation data suggests a stronger lower bound, but we would most likely require a different approach.

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