On the Galois Correspondence between convering spaces of a space and subgroups of its fundamental group

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Abstract

Our goal will be to state the classification theorem for the covering spaces of a path-connected, locally path connected and semilocally simply connected space, and apply it in some concrete situations. Our main driving examples will be S^1 and $S^1 \vee S^1$.

Basic Definitions

Throughout the poster, a **map** will mean a continuous map. Let X be any topological space. A **path** in X is a map $f : I \to X$. A **homotopy** of paths (with fixed endpoints) in X is a family $f_t : I \to X$ of paths for $t \in I$ such that the following hold.

1. The endpoints $f_t(0)$ and $f_t(1)$ are *independent* of t.
2. The associated map $F : I \times I \to X$ defined by $F(s, t) = f_t(s)$ is continuous.

In this case, we write $f_0 \simeq f_1$.

Proposition

Homotopy of paths with fixed basepoints is an equivalence relation. The equivalences classes are called **homotopy classes**.

If f, g are any paths in X with $f(1) = g(0)$, we define the **product** $f \cdot g$ of these paths as the path

$$f \cdot g(s) = \begin{cases} f(2s) & ,0 \le s \le 1/2 \\ g(2s-1) & ,1/2 \le s \le 1 \end{cases}$$

These tools allow us to define the **fundamental group**.

Fundamental Group

The set of all homotopy classes of **loops** in X on a **basepoint** $x_0 \in X$ is a group with respect to the multiplication $[f] \cdot [g] = [f \cdot g]$, where $[\cdot]$ represents the homotopy class of a loop. This group is denoted $\pi_1(X, x_0)$

Fundamental Group as a Functor

Suppose $\varphi : (X, x_0) \to (Y, y_0)$ is a map with $\varphi(x_0) = y_0$. Then, via the map φ, we can send loops in X to loops in Y. More specifically, given a loop f in X based at x_0, φf (composition) is a loop in Y based at y_0.

Proposition

For a map $\varphi : (X, x_0) \to (Y, y_0)$ as above, there is a group homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $\varphi_*([\gamma]) = [\varphi\gamma]$ for any loop γ in X based at x_0. This is called the homomorphism **induced by** φ. If $1 : X \to X$ is the identity map, then 1_* is the identity homomorphism on $\pi_1(X, x_0)$. Moreover, given a composition $(X, x_0) \xrightarrow{\varphi} (Y, y_0) \xrightarrow{\psi} (Z, z_0)$, the map $\psi\varphi : X \to Z$ satisfies

$$(\psi\varphi)_* = \psi_*\varphi_*$$

So, the fundamental group can be viewed as a **functor** from the category of topological spaces to the category of groups.

This gives a great example of converting information about topological spaces to information about groups.

Deformation Retractions

Let X be a space, and let $A \subset X$. A map $r : X \to A$ is said to be a *retraction* if $r|_A = \text{id}$. Further, r is said to be a (weak) *deformation retraction* if there is some homotopy r_t with $r_0 = 1$, $r_t|_A = \text{id}$ for every t and $r_1 = r$. It can be shown that if r is a deformation retraction, then if $i : A \hookrightarrow X$ is the inclusion map then $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is an *isomorphism*.

Deformation retractions are some of the most useful homotopies as they can *simplify* our work while computing fundamental groups. For instance, they can be used to compute fundamental groups of graphs along with a tool known as **Van Kampen's Theorem**.

Covering Spaces

Let X be any topological space. A space $\tilde{X}$ is called a **covering space** of X if there is a map $p : \tilde{X} \to X$ with the following property: every $x \in X$ has an open neighborhood $U_x \subset X$ such that $p^{-1}(U_x)$ is a disjoint union of sets in $\tilde{X}$, each of which is *homeomorphic* to U_x under p. Each set in the disjoint union is called a **sheet**.

Lifing Paths (Proposition)

The most useful property of covering spaces is **path lifting**; given any path f in X starting at x_0 and given any $\tilde{x_0} \in p^{-1}(x_0)$, there is a *unique* path $\tilde{f}$ in $\tilde{X}$ starting at $\tilde{x_0}$ with $f = p\tilde{f}$. A version of *homotopy lifting* is also true: given any homotopy $f_t : I \to X$ of paths starting at x_0 and $\tilde{x_0} \in p^{-1}(x_0)$, there is a *unique* homotopy $\tilde{f_t} : I \to \tilde{X}$ of paths starting at $\tilde{x_0}$ with $f_t = p\tilde{f_t}$ for each t.

An example: S^1

One of the best examples of covering spaces is given by the **helix** in $\mathbb{R}^3$, which is the set $\{(\cos t, \sin t, t) \mid t \in \mathbb{R}\}$. By projecting onto the xy plane, this becomes a covering space for the circle S^1. By just using this information along with the *lifting properties*, we can show that $\pi_1(S^1) = \langle[\omega]\rangle$, where ω is the loop that winds around S^1 *once*. So, we can conclude that $\pi_1(S^1) \cong \mathbb{Z}$, a very important calculation.

The Galois Correspondence

Our driving example will be the **wedge sum** $S^1 \vee S^1$, which is two circles attached at a single point. We will also assume in this section that all spaces are path-connected. A basic fact about a **covering map** $p : \tilde{X} \to X$ is that the induced homomorphism p_* is *injective*, i.e $\pi_1(\tilde{X})$ is a *subgroup* of $\pi_1(X)$. We ask the opposite question: does every subgroup of $\pi_1(X)$ correspond to some covering space of X? The answer is yes, provided that X is nice locally.

Theorem

Let (X, x_0) be any path-connected, locally path connected and **semi-locally simply connected** space. Then for **every** subgroup H of $\pi_1(X, x_0)$, there is a covering space $(X_H, \tilde{x_0})$ with a covering map $p_H : (X_H, \tilde{x_0}) \to (X, x_0)$ such that

$$(p_H)_*(\pi_1(X_H, \tilde{x_0})) = H$$

Theorem(contd')

More specifically, there is a **simply-connected** covering space $(\tilde{X}, \tilde{x_0})$ of (X, x_0) (i.e $(\tilde{X}, \tilde{x_0})$ has trivial fundamental group) and every other X_H can be constructed as a *quotient space* of $\tilde{X}$. In this case $\tilde{X}$ is called the **universal cover**.

A good way to look at this correspondence is using **lattices**: if the lattice of subgroups of $\pi_1(X)$ is given, then turning this lattice upside down gives us the lattice of covering spaces of X corresponding to subgroups of $\pi_1(X)$.

Examples: S^1 and $S^1 \vee S^1$

We know that $\pi_1(S^1) \cong \mathbb{Z}$, and as we saw earlier that the universal cover of S^1 is the **helix**. The lattice of $\mathbb{Z}$ is fairly easy to understand; every subgroup is of the form $n\mathbb{Z}$. So the correspondence theorem above tells us that this lattice inverted is the lattice of covering spaces of S^1, with the **helix** sitting at the bottom of the lattice.

Now we consider $S^1 \vee S^1$. **Van Kampen's Theorem** can be used to conclude that $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$, the free product of two $\mathbb{Z}$'s. It is easy to see that $\mathbb{Z} * \mathbb{Z}$ is just the *free group* $\langle a, b\rangle$ on two generators. Determining the universal cover for $S^1 \vee S^1$ requires more work; we will use the so called **Cayley-Graph** of the free group $\langle a, b\rangle$ on two generators, and the graph will be the universal cover for $S^1 \vee S^1$. The vertices of the graph will be all the elements of the free group $\langle a, b\rangle$. If $g \in \langle a, b\rangle$, then there will be edges connecting g to ga and gb (i.e we connect g to $g\alpha$ for every generator α of the group). This gives us the following picture, in which the origin represents the identity e.