Conservative Balls and Projections

Def: (Conservative Balls) Let $\hat{\theta}$ be the fixed strategy. The conservative ball $B(\hat{\theta}, r_t)$ at time t is defined as follows.

$$(+)\quad r_t := \left[1 - \left(\frac{L_{t-1} - (1+\alpha)\tilde{L}_{t-1} - \alpha\varepsilon_\ell + 1}{DG}\right)^+\right] D$$

where:

$$a^+ := \max(0, a)$$

$$D := \text{diameter of } \Theta = \sup_{x,y \in \Theta} \|x - y\|$$

$$G := \sup_{x \in \Theta} \|\nabla f_t(x)\|$$

Q: Why the name "conservative ball"

Ans: Playing choice $\theta_t \in B(\hat{\theta}, r_t)$ ensures that the budget $Z_t(U) = (1+\alpha)\tilde{L}_t - L_t$ is non-negative, which we will now prove.

THM: Suppose the budget $Z_{t-1}(U)$ at time $t-1$ satisfies the following.

$$Z_{t-1}(U) \geq 0 \quad \left(\text{i.e. } (1+\alpha)\tilde{L}_t \geq L_t\right)$$

Then, any choice of θ_t s.t. $\theta_t \in B(\hat{\theta}, r_t)$

ensures that the budget at time t is non-negative, ie

$$Z_t(u) \geq 0.$$

Pf: By the convexity of f_t, we have the following.

$$f_t(\hat{\theta}) \geq f_t(\theta_t) + \langle \nabla f_t(\theta_t), \hat{\theta} - \theta_t \rangle$$

$$\Rightarrow \quad f_t(\theta_t) - f_t(\hat{\theta}) \leq \langle \nabla f_t(\theta_t), \theta_t - \hat{\theta} \rangle$$

Add $-\alpha f_t(\hat{\theta})$ to both sides to get the following.

$$f_t(\theta_t) - (1+\alpha) f_t(\hat{\theta}) \leq \langle \nabla f_t(\theta_t), \theta_t - \hat{\theta} \rangle - \alpha f_t(\hat{\theta})$$

$$\leq \langle \nabla f_t(\theta_t), \theta_t - \hat{\theta} \rangle - \alpha \varepsilon_\ell$$

$$\leq \| \nabla f_t(\theta_t) \| \cdot \| \theta_t - \hat{\theta} \| - \alpha \varepsilon_\ell \quad \text{(C.S Ineq.)}$$

$$\leq g \cdot r_t - \alpha \varepsilon_\ell \quad \text{(by assumption)}$$

Call the last inequality (Δ).

By definition of r_t,

$$0 \le r_t \le D.$$

Case 1: $r_t < D$.

$$\Rightarrow r_t = \frac{(1+\alpha)\tilde{L}_{t-1} - L_{t-1} + \alpha\varepsilon_e}{G}$$

Plug r_t in (Δ) to get the following.

$$f_t(\theta_t) - (1+\alpha) f_t(\hat{\theta})$$

$$\le G r_t - \alpha\varepsilon_e$$

$$= (1+\alpha)\tilde{L}_{t-1} - L_{t-1}$$

$$\Rightarrow (1+\alpha)\tilde{L}_t - L_t \ge 0 \quad \text{(Proved)}$$

Case 2: $r_t = D$. In this case,

$$\frac{L_{t-1} - (1+\alpha)\tilde{L}_{t-1} - \alpha\varepsilon_e}{GD} + 1 \le 0$$

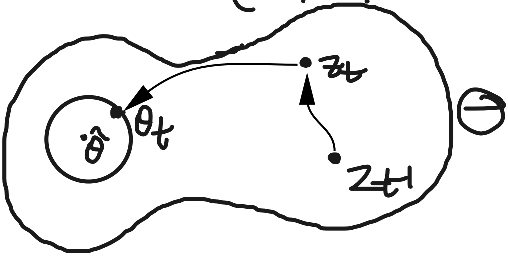
$$\Rightarrow L_{t-1} - (1+\alpha)\tilde{L}_{t-1} - \varepsilon\alpha_e + GD \le 0$$

$$\Rightarrow GD - \varepsilon\alpha_e \le (1+\alpha)\tilde{L}_{t-1} - L_{t-1}$$

From here, same as Case 1. Q.E.D

$\longrightarrow$

Conservative Projection: Given $z_t \in \Theta$, project it onto $B(\hat{\theta}, r_t)$.



$$\theta_t = \Pi_{B(\hat{\theta}, r_t)}(z_t) = \beta_t \hat{\theta} + (1-\beta_t) z_t$$

$\Pi_{B(\hat{\theta}, r_t)}$ → "Projection"

$$\beta_t = \begin{cases} 0, & \text{if } z_t \in B(\hat{\theta}, r_t) \\ 1 - \dfrac{r_t}{\|z_t - \hat{\theta}\|}, & \text{o/w} \end{cases}$$

Remark: Projection is efficient (in fact, we have found a closed form for it). However, note that it is efficient only because we are dealing with balls.